a) By induction, \( \text{dim } \text{Sym}_n = \frac{n(n+1)}{2} \)

Note: If \( A \in \text{M}_{n \times n}(\mathbb{R}) \) is symmetric, then knowing only the upper (or lower) triangular entries determines the entire matrix. There are \( \frac{n(n+1)}{2} \) such entries.

\[
A = \begin{pmatrix}
1 & & & & \\
& 2 & & & \\
& & 3 & & \\
& & & \ddots & \\
& & & & & n
\end{pmatrix}
\]

\[
1 + 2 + \ldots + n = \frac{n(n+1)}{2}
\]

b) By induction, \( \text{dim } \text{Skew}_n = \frac{n(n-1)}{2} \).

Note: If \( A \in \text{M}_{n \times n}(\mathbb{R}) \) is skew, then it has zeroes along the diagonal, and knowing only the entries above (or below) the diagonal determines the entire matrix. There are \( \frac{n(n-1)}{2} \) such entries.

\[
\begin{pmatrix}
0 & & & & \\
& 0 & & & \\
& & 0 & & \\
& & & \ddots & \\
& & & & 0
\end{pmatrix}
\]

c) \( \Phi : \text{M}_{n \times n}(\mathbb{R}) \to \text{M}_{n \times n}(\mathbb{R}) \), \( \Phi(A) = AQ + QA^t \) is linear.

Moreover, \( \text{Im } \Phi \subseteq \text{Sym}_n \), since \( \forall A \in \text{M}_{n \times n}(\mathbb{R}) \), we have

\[
\Phi(A)^t = (AQ + QA^t)^t = QA^t + A^tQ^t = QA + AQ = \Phi(A)
\]

\( Q \in \text{Sym}_n \)

To show that \( \forall M \in \text{Sym}_n \exists A \in \text{M}_{n \times n}(\mathbb{R}) \) such that \( \Phi(A) = M \), we need to show \( \text{Im } \Phi = \text{Sym}_n \).
For this, since we already know \( \text{Im} \phi \subset \text{Sym}_n \), it suffices to show \( \dim \ker \phi = \dim \text{Sym}_n \). By the Rank Nullity Theorem,

\[
\dim \ker \Phi + \dim \text{Im} \phi = \dim \text{M}_{nxn}(\mathbb{R}) = n^2.
\]

so \( \dim \text{Im} \phi = n^2 - \dim \ker \Phi \).

Note that there is an isomorphism

\[
\Phi : \text{Skew}_n \longrightarrow \ker \Phi
\]

\[
X \mapsto XQ^{-1}
\]

Indeed, \( \Phi(X) = XQ^{-1} \in \ker \Phi \) since we can compute

\[
\Phi(XQ^{-1}) = (XQ^{-1})Q + Q(XQ^{-1})^t
\]

\[
= XQ^{-1}Q + Q(Q^{-1})^tX^t
\]

\[
= X + Q(Q^{-1})^tX^t
\]

\[
= X + X^t = 0 \quad \text{for all } X \in \text{Skew}_n.
\]

Clearly \( \Phi \) is invertible, \( \Phi^{-1}(Y) = YQ \). In particular, since \( \ker \Phi \) and \( \text{Skew}_n \) are isomorphic, they have the same dimension, hence \( \dim \ker \Phi = \dim \text{Skew}_n = \frac{n(n-1)}{2} \).

Thus \( \dim \text{Im} \Phi = n^2 - \dim \ker \Phi \)

\[
= n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2} = \dim \text{Sym}_n,
\]

so \( \text{Im} \Phi = \text{Sym}_n \), concluding the proof. \( \square \)