HW 6 #4

a) By induction, $\dim \mathrm{Sym}_n = \dfrac{n(n+1)}{2}$

<u>Note</u>: If $A \in M_{n\times n}(\mathbb{R})$ is <u>symmetric</u>, then knowing only the upper (or lower) triangular entries determines the entire matrix. There are $\dfrac{n(n+1)}{2}$ such entries.

$$A = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet \\ & & & & & \bullet \end{pmatrix} \begin{array}{l} \text{— } n \\ \vdots \\ \text{— } 4 \\ \text{— } 3 \\ \text{— } 2 \\ \text{— } 1 \; + \end{array}$$

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

b) By induction, $\dim \mathrm{Skew}_n = \dfrac{n(n-1)}{2}$.

<u>Note</u>: If $A \in M_{n\times n}(\mathbb{R})$ is <u>skew</u>, then it has zeros along the diagonal, and knowing only the entries above (or below) the diagonal determines the entire matrix. There are $\dfrac{n(n-1)}{2}$ such entries.

$$\begin{pmatrix} 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 0 & \bullet & \bullet & \bullet & \bullet \\ & & & 0 & \bullet & \bullet & \bullet \\ & & & & 0 & \bullet & \bullet \\ & & & & & 0 & \bullet \\ & & & & & & 0 \end{pmatrix}$$

c) $\Phi : M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$, $\Phi(A) = AQ + QA^t$ is linear.

Moreover, $\mathrm{Im}\,\Phi \subset \mathrm{Sym}_n$, since $\forall A \in M_{n\times n}(\mathbb{R})$, we have

$$\Phi(A)^t = (AQ + QA^t)^t = Q^tA^t + AQ^t \underset{\substack{\uparrow \\ Q \in \mathrm{Sym}_n}}{=} QA^t + AQ = \Phi(A).$$

To show that $\forall M \in \mathrm{Sym}_n \; \exists A \in M_{n\times n}(\mathbb{R})$ such that $\Phi(A) = M$, we need to show $\mathrm{Im}\,\Phi = \mathrm{Sym}_n$.

For this, since we already know $\operatorname{Im} \phi \subset \operatorname{Sym}_n$, it suffices to show $\dim \operatorname{Im} \phi = \dim \operatorname{Sym}_n$. By the Rank Nullity Theorem,

$$\dim \operatorname{Ker} \Phi + \dim \operatorname{Im} \phi = \dim M_{n \times n}(\mathbb{R}) = n^2.$$

so $\dim \operatorname{Im} \Phi = n^2 - \dim \operatorname{Ker} \Phi$.

Note that there is an isomorphism

$$\Psi : \operatorname{Skew}_n \longrightarrow \operatorname{Ker} \Phi$$
$$X \longmapsto XQ^{-1}$$

Indeed, $\Psi(X) = XQ^{-1} \in \operatorname{Ker} \Phi$ since we can compute

$$\begin{aligned}
\Phi(XQ^{-1}) &= (XQ^{-1})Q + Q(XQ^{-1})^t \\
&= X \cancel{Q^{-1}Q} + Q(Q^{-1})^t X^t \\
&= X + \cancel{QQ^{-1}} X^t \\
&= X + X^t = 0 \quad \text{for all } X \in \operatorname{Skew}_n.
\end{aligned}$$

Clearly Ψ is invertible, $\Psi^{-1}(Y) = YQ$. In particular, since $\operatorname{Ker} \Phi$ and Skew_n are isomorphic, they have the same dimension: $\dim \operatorname{Ker} \Phi = \dim \operatorname{Skew}_n = \frac{n(n-1)}{2}$.

Thus
$$\begin{aligned}
\dim \operatorname{Im} \Phi &= n^2 - \dim \operatorname{Ker} \Phi \\
&= n^2 - \frac{n(n-1)}{2} \\
&= \frac{n(n+1)}{2} = \dim \operatorname{Sym}_n,
\end{aligned}$$
so $\operatorname{Im} \Phi = \operatorname{Sym}_n$ concluding the proof. $\square$