Math465/501 Spring 2018 Homework Set 2

1. Shifrin (P31) Exercise 1

(a) Proof: Suppose without loss of generality $x=(0,0,1)$, $y=(\sin u_0, 0, \cos u_0)$

Let $\alpha = (\sin u(t)\cos v(t), \sin u(t)\sin v(t), \cos u(t))$ $\quad a \le t \le b$, with $\alpha(a)=0$,

$v(a)=0$, $u(b)=u_0$, $v(b)=0$

For simpler notation, $u=u(t)$, $v=v(t)$, $u'=u'(t)$, $v'=v'(t)$

$\alpha' = (\cos(u)\cos(v)u' - \sin(u)\sin(v)v', \cos(u)\sin(v)u' + \sin(u)\cos(v)v', -\sin(u)u'$.

Since $|\alpha'| \ge 0$, arclength $\int_a^b |\alpha'|dt$ is minimized when $\int_a^b |\alpha'|^2 dt$ is minimized.

$|\alpha'|^2 = \cos^2(u)\cos^2(v)u'^2 + \sin^2(u)\sin^2(v)v'^2 - \cancel{2\cos(u)\sin(u)\cos(v)\sin(v)u'v'}$

$+\cos^2(u)\sin^2(v)u'^2 + \sin^2(u)\cos^2(v)v'^2 + \cancel{2\cos(u)\sin(u)\sin(v)\cos(v)u'v'} + \sin^2(u)u'^2$

$= \cos^2(u)u'^2 + \sin^2(u)v'^2 + \sin^2(u)u'^2$

$= (u')^2 + \sin^2(u)v'^2$

We want to minimize $\int_a^b (u'(t))^2 + \sin^2(u(t))(v'(t))^2 dt$. We can reduce this problem to minimizing: $\int_0^T u'(t)^2 + \sin^2(u(t))v'(t)^2 dt$. subject to the boundary conditions:

$$\begin{cases} u(0)=0 & v(0)=0 \\ u(T)=u_0>0 & v(T)=0. \end{cases} \quad (BC)$$

Notice that both terms in the integral are non-negative, hence it is equivalent to minimizing $\int_0^T u'(t)^2 dt$ and $\int_0^T \sin^2(u(t))v'(t)^2 dt$ seperately.

Let u_s, v_s be 1-parameter variations of u and v respectively, and write $\dot{u} = \frac{d}{ds}u_s\big|_{s=0}$, $\dot{v} = \frac{d}{ds}u_s\big|_{s=0}$. Then we find minimizers using the first variation:

$$\frac{d}{ds}\Big|_{s=0}\int_0^T u_s'(t)^2 dt = \int_0^T 2u'\dot{u}' dt \overset{\text{parts}}{=} -\int_0^T 2u''\dot{u}\,dt = 0 \quad \forall \dot{u} \iff u'' \equiv 0 \overset{(BC)}{\iff} \boxed{u(t) = \frac{u_0}{T}t} \quad ①$$

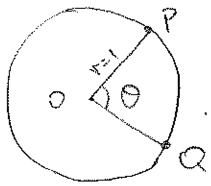
$$\frac{d}{ds}\Big|_{s=0}\int_0^T \sin^2(u_s(t))v_s'(t)^2 dt = \int_0^T 2\sin(u)\cos(u)(v')^2\dot{u} + 2\sin(u)^2 v'\dot{v}' dt = 0 \quad \forall \dot{u},\dot{v}$$

$$\iff \begin{cases} \sin(2u)v'=0 \\ \sin(u)^2 v'=0 \end{cases} \Rightarrow v'=0 \text{ a.e.} \overset{(BC)}{\Longrightarrow} \boxed{v(t)\equiv 0} \quad ② \quad \text{(we integrat } v \text{ using } v'=0 \text{ a.e.)}$$

($\sin(u)\ne 0$ except at finitely many points)

Thus $\alpha(t)$ must be a great circle. $\square$

(b) The shortest path between P and Q is an arc of a great circle connecting them by part (a):



$P\cdot Q = |OP||OQ|\cos\theta \Rightarrow \theta = \arccos(P\cdot Q)$.

$\text{Arclength}(PQ) = 2\pi r\cdot\frac{\theta}{2\pi} = \arccos(P\cdot Q)$.

2. Shifrin (page 33) Exercise 12.

Proof: Let $\alpha: [0, L] \to \Sigma$ be the arclength parametrization of Γ, define $F: [0,L] \times [0, 2\pi) \to \Sigma$ by $F(s,\phi) = \xi$ where $\xi^{\perp}$ is the great circle making angle ϕ with Γ at $\alpha(s)$. Let t denotes the number of times F take the value ξ. Fix ξ on the sphere, then if F takes the value ξ, then there is a point (s,ϕ) corresponding to $\alpha(s)$ s.t $F(s,\phi) = \xi$, hence $t \le \#(\Gamma \cap \xi^{\perp})$. Now consider the points in $\Gamma \cap \xi^{\perp}$, for each intersecting point, even if the angle ϕ are the same, the "time" s will be different, hence for each $\alpha(s) \in \Gamma \cap \xi^{\perp}$, the angle is ϕ and $F(s,\phi) = \xi$ (change in s avoids double counting). Hence $t \ge \#(\Gamma \cap \xi^{\perp})$.

Thus $t = \#(\Gamma \cap \xi^{\perp})$. So F is a "multi-parametrization" of Σ and we have that $\int_{\Sigma} \#(\Gamma \cap \xi^{\perp}) d\xi = \int_0^L \int_0^{2\pi} \left\| \frac{\partial F}{\partial s} \times \frac{\partial F}{\partial \phi} \right\| d\phi\, ds$.

Let $v(s,\phi) = \cos\phi T(s) + \sin\phi(\alpha(s) \times T(s))$, $v(s,\phi) \cdot \alpha(s) = \cos\phi T(s) \cdot \alpha(s) + \sin\phi(\alpha(s) \times T(s)) \cdot \alpha(s)$

$$= 0 - \sin\phi\,(\alpha(s) \times \alpha(s)) \cdot T(s))$$
$$= 0.$$

Thus $v(s,\phi)$ is the tangent vector to the great circle.

Then geometrically we can deduce that $F(s,\phi) = \xi = \alpha(s) \times v(s,\phi)$.

$$\frac{\partial F}{\partial \phi} = \alpha(s) \times \frac{\partial v}{\partial \phi} = \alpha(s) \times (-\sin\phi T(s) + \cos\phi(\alpha(s) \times T(s))) = -\sin\phi(\alpha(s) \times T(s)) + \cos\phi T$$
$$= -v(s,\phi)$$

$$\frac{\partial F}{\partial s} = \alpha'(s) \times v(s,\phi) + \alpha(s) \times \frac{\partial v}{\partial s}.$$

Notice that $\alpha(s) \times \frac{\partial v}{\partial s} = \alpha(s) \times (\cos\phi T'(s) + \sin\phi(\alpha'(s) \times T(s) + \alpha(s) \times T'(s))$.

$$= \alpha(s) \times (\cos\phi (T'(s)) + \sin\phi(\alpha(s) \times T'(s))).$$
$$= \cos\phi\, \alpha(s) \times T'(s) + \alpha(s) \times \sin(\phi)(\alpha(s) \times T'(s))$$
$$= \cos\phi\, T(s) - \sin\phi(T(s) \times \alpha(s)$$
$$= \cos\phi T(s) + \sin\phi(\alpha(s) \times T(s)) = v.$$

Then $\frac{\partial F}{\partial \phi} \times \frac{\partial F}{\partial s} = \frac{\partial F}{\partial \phi} \times (\alpha'(s) \times v(s,\phi))$.

Hence $\left\| \frac{\partial F}{\partial \phi} \times \frac{\partial F}{\partial s} \right\| = \left\| \frac{\partial F}{\partial \phi} \times [\alpha'(s) \times v(s,\phi)] \right\|$

$$= \| (-v) \times (\alpha'(s) \times (\cos\phi T(s) + \sin(\phi)(\alpha(s) \times T(s)))) \|$$

$$= |\sin\phi| \, \| \alpha(s) \times v(s,\phi) \|$$

$$= |\sin\phi|$$

Then we have:

$$\int_{\Sigma} \#(\Gamma \cap \xi^{\perp}) \, d\xi = \int_0^L \int_0^{2\pi} |\sin\phi| \, d\phi \, ds$$

$$= \int_0^L \left(\int_0^{\pi} \sin\phi \, d\phi + \int_{\pi}^{2\pi} -\sin\phi \, d\phi \right) ds$$

$$= \int_0^L \left(-\cos\phi \Big|_0^{\pi} + \cos\phi \Big|_{\pi}^{-2\pi} \right) ds$$

$$= \int_0^L 2 + 2 \, ds$$

$$= 4L.$$

3. Shifrin (P41) Exercise 2

$\alpha(t) = X(u(t), v(t))$ $a \le t \le b$, $\alpha'(t) = u'(t)\, X_u(u(t), v(t)) + v'(t)\, X_v(u(t), v(t))$.

$$\text{length}(\alpha) = \int_a^b |\alpha'(t)|\, dt = \int_a^b \sqrt{\alpha'(t)\cdot\alpha'(t)}\, dt$$

$$= \int_a^b \sqrt{I_{\alpha(t)}(\alpha'(t), \alpha'(t))}\, dt$$

For simpler notation, write $\alpha'(t)$ as $\alpha' = u' X_u + v' X_v$.

Then $\text{length}(\alpha) = \int_a^b \sqrt{I_\alpha(\alpha', \alpha')}\, dt$

$$= \int_a^b \sqrt{(u'X_u + v'X_v)\cdot(u'X_u + v'X_v)}\, dt$$

$$= \int_a^b \sqrt{(u')^2(X_u\cdot X_u) + u'v' X_u\cdot X_v + v'u' X_v\cdot X_u + (v')^2(X_v\cdot X_v)}\, dt$$

$$= \int_a^b \sqrt{(u')^2(X_u\cdot X_u) + 2u'v'(X_u\cdot X_v) + (v')^2(X_v\cdot X_v)}\, dt$$

$$= \int_a^b \sqrt{E(u(t), v(t))(u'(t))^2 + 2F(u(t), v(t))\, u'(t) v'(t) + G(u(t), v(t))(v'(t))^2}\, dt.$$

Let $\alpha \subset M$ and $\alpha^* \subset M^*$ be corresponding paths that in locally isometric surfaces, then $E = E^*$, $G = G^*$, $F = F^*$ where E^*, G^* and F^* corresponds to $\alpha^* \subset M^*$ parametrized by $u(t), v(t)$ for $t \in [a,b]$.

Then $\text{length}(\alpha^*) = \int_a^b \sqrt{E^*(u(t), v(t))(u'(t))^2 + 2F^*(u(t), v(t))u'(t)v'(t) + G^*(u(t), v(t))(v'(t))^2}\, dt$

$$= \int_a^b \sqrt{E(u(t), v(t))(u'(t))^2 + 2F(u(t), v(t))u'(t)v'(t) + G(u(t), v(t))(v'(t))^2}\, dt$$

$= \text{length}(\alpha)$.

$\square$

3. Shifrin (P41) 4 (a)

$$X(u,v) = ((a+b\cos u)\cos v,\ (a+b\cos u)\sin v,\ b\sin u) \qquad 0 \le u, v \le 2\pi$$

$$X_u = (-b\sin u\cos v,\ -b\sin u\sin v,\ b\cos u)$$

$$X_v = (-(a+b\cos u)\sin v,\ (a+b\cos u)\cos v,\ 0).$$

$$E = X_u \cdot X_u = b^2\sin^2 u\cos^2 v + b^2\sin^2 u\sin^2 v + b^2\cos^2 u$$

$$= b^2\sin^2 u + b^2\cos^2 u = b^2$$

$$F = X_u \cdot X_v = \cancel{ab\sin u\sin v\cos v} + \cancel{b^2\sin u\cos u\sin v\cos v}$$

$$- \cancel{ab\sin u\sin v\cos v} - \cancel{b^2\sin u\cos u\sin v\cos v} = 0.$$

$$G = X_v \cdot X_v = (a+b\cos u)^2\sin^2 v + (a+b\cos u)^2\cos^2 v$$

$$= (a+b\cos u)^2$$

$$\text{Area} = \int_0^{2\pi}\int_0^{2\pi} \| X_u \times X_v \|\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} \sqrt{EG - F^2}\, du\, dv.$$

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{b^2(a+b\cos u)^2}\, du\, dv$$

$$= \int_0^{2\pi} 2\pi ab.\, dv$$

$$= 4\pi^2 ab.$$

3. Shifrin (P41) 4(c).

$X(u,v) = a(\sin u \cos v, \sin u \sin v, \cos u)$ $\qquad 0 \le u_0 \le u \le u_1 \le \pi$. $\quad 0 \le v \le 2\pi$.

$X_u = a(\cos u \cos v, \cos u \sin v, -\sin u)$.

$X_v = a(-\sin u \sin v, \sin u \cos v, 0)$.

$$E = X_u \cdot X_u = a^2(\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u)$$
$$= a^2(\cos^2 u + \sin^2 u)$$
$$= a^2.$$

$$F = X_u \cdot X_v = a^2(\cancel{-\sin u \cos u \sin v \cos v} + \sin u \cos u \sin v \cos v) = 0.$$

$$G = X_v \cdot X_v = a^2(\sin^2 u \sin^2 v + \sin^2 u \cos^2 v)$$
$$= a^2 \sin^2 u.$$

$$\text{Area} = \int_0^{2\pi}\int_{u_0}^{u_1} \|X_u \times X_v\| \, du\,dv = \int_0^{2\pi}\int_{u_0}^{u_1} \sqrt{EG-F^2}\, du\,dv.$$

$$= \int_0^{2\pi}\int_{u_0}^{u_1} \sqrt{a^4 \sin^2 u}\, du\,dv.$$

$$= a^2 \int_0^{2\pi} \cos u_0 - \cos u_1 \, dv$$

$$= 2\pi a^2(\cos u_0 - \cos u_1).$$

4. Shifrin (P42) 5

Proof: Without loss of generality, suppose all normal lines pass through the origin. Let n be the unit normal vector, then for the surface $X(u,v)$, there is function $\lambda(u,v)$ s.t $X(u,v) = \lambda(u,v) n$

Since n is normal $X_u \cdot n = X_v \cdot n = 0$.

Then $X \cdot X_u = \lambda(u,v) n \cdot X_u = 0$

and $X \cdot X_v = \lambda(u,v) n \cdot X_v = 0$.

Notice that $(X \cdot X)_u = X_u \cdot X + X \cdot X_u = 2X \cdot X_u = 0$

and $(X \cdot X)_v = X_v \cdot X + X \cdot X_u = 2X \cdot X_u = 0$.

Thus $X \cdot X = \|X\|^2$ is constant, hence $\|X\|$ is constant.

Therefore, the surface $X(u,v)$ is a portion of a sphere.

$\square$

4 Shifrin (P42) 6

Proof: For vectors w, z in $\mathbb{R}^2$, we can write them in terms of the standard basis e_1, e_2: $w = ae_1 + be_2$, $z = ce_1 + de_2$ for $a,b,c,d \in \mathbb{R}$.

The vectors corresponding to w, z in the surface are:

$w_s = dx(u,v)\, w$ and $z_s = dx(u,v)\, z$.

Then $X(u,v)$ is conformal $\iff \dfrac{I(w_s, z_s)}{\sqrt{I(w_s,w_s)\, I(z_s,z_s)}} = \dfrac{w \cdot z}{|w||z|}$

Now we can prove our claim:

($\Rightarrow$) Let $w = (1,0)$, $z = (0,1)$. Since $X(u,v)$ is conformal, we have that $I(w_s, z_s) = 0$ $(w \cdot z = 0)$.

Also $w_s = dx(u,v) w = dx(u,v)(1,0) = X_u$.

$z_s = dx(u,v) z = dx(u,v)(0,1) = X_v$

Then $0 = I(w_s, z_s) = I(X_u, X_v) = F$.

Notice that $(w+z) = (1,1)$ and $(w-z) = (1,-1)$ and $(w+z)\cdot(w-z) = 0 \implies (w+z) \perp (w-z)$.

Since $X(u,v)$ is conformal, we have:

$0 = I(w_s + z_s, w_s - z_s) = I(w_s, w_s) - I(z_s, z_s)$

$= I(X_u, X_u) - I(X_v, X_v)$

$= E - G = 0$

This implies that $E = G$.

($\Leftarrow$) Suppose $E = G$ and $F = 0$

Recall that we identify $w = (a,b)$ in $\mathbb{R}^2$ with $w_s = aX_u + bX_v$ in the surface, we identify $z = (c,d)$ in $\mathbb{R}^2$ with $z_s = cX_u + bX_v$ in the surface.

$$I(w_s, z_s) = acE + F(ad+bc) + Gbd = E(ac+bd)$$
$$= E\, w \cdot z.$$

$I(w_s, w_s) = E\, w \cdot w$ and $I(z_s, z_s) = E\, z \cdot z$

Then
$$\frac{I(w_s, z_s)}{\sqrt{I(w_s,w_s)\, I(z_s,z_s)}} = \frac{E\, w\cdot z}{\sqrt{E^2\, w\cdot w\; z\cdot z}} = \frac{w \cdot z}{|w|\,|z|}.$$

Therefore $X(u,v)$ is conformal $\square$

4. Shifrin (P42) Exercise 7.

Claim: A parametrization preserves area and is conformal $\iff$ it is a local isometry.

Proof: ($\Rightarrow$) Suppose X is such a surface. By Problem 6, the parametrization is conformal if and only if $E=G$ and $F=0$.

Since the parametrization preserves area, by the area formula, we need $\sqrt{EG-F^2}=1 \Rightarrow E=G=1$

For the parametrization uv-plane, $y(u,v)=(u,v)$, $Y_u=(1,0)$, $Y_v=(0,1)$, then $E^*=Y_u\cdot Y_u=1=G^*=Y_v\cdot Y_v=1$ and $F^*=0$

Since $E=E^*$, $G=G^*$ and $F=F^*$, this is a local isometry.

($\Leftarrow$). Suppose the parametrization is a local isometry, by the same argument above, $E=G=1$ and $F=0$ for the surface. Hence by Problem 6, the parametrization is conformal. Moreover, $\sqrt{EG-F^2}=1$ implies that the parametrization preserves area.

$\square$

5. (a) Claim: Gauss curvature: $\dfrac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2} = \dfrac{\det(\mathrm{Hess}(f))}{(1+|\mathrm{grad}\, f|^2)^2}$

Mean Curvature: $\dfrac{f_{uu}(1+f_v^2) - 2f_{uv}f_uf_v + f_{vv}(1+f_u^2)}{2(1+f_u^2+f_v^2)^{3/2}} = \dfrac{f_{uu}(1+f_v^2) - 2f_{uv}f_uf_v + f_{vv}(1+f_u^2)}{2(1+|\mathrm{grad}\, f|^2)^{3/2}}$

Computation: $M(u,v) = (u, v, f(u,v))$.

$M_u = (1, 0, f_u)$ $\quad M_v = (0, 1, f_v)$.

$M_u \times M_v = (-f_u, -f_v, 1)$

$N = \dfrac{M_u \times M_v}{\|M_u \times M_v\|} = \dfrac{(-f_u, -f_v, 1)}{\sqrt{f_u^2+f_v^2+1}}$

$M_{uu} = (0, 0, f_{uu})$ $\quad M_{vv} = (0, 0, f_{vv})$ $\quad M_{uv} = (0, 0, f_{uv})$.

First fundamental form coefficients:

$E = M_u \cdot M_u = f_u^2+1$ $\quad F = M_u \cdot M_v = f_uf_v$ $\quad G = M_v \cdot M_v = 1+f_v^2$

Second fundamental form coefficients:

$e = N \cdot M_{uu} = \dfrac{1}{\sqrt{f_u^2+f_v^2+1}} f_{uu}$

$f = N \cdot M_{uv} = \dfrac{1}{\sqrt{f_u^2+f_v^2+1}} f_{uv}$

$g = N \cdot M_{vv} = \dfrac{1}{\sqrt{f_u^2+f_v^2+1}} f_{vv}$.

$S_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ is the shape operator,

then $\boxed{\text{Gauss Curvature} = \det(S_p) = \dfrac{eg-f^2}{EG-F^2} = \dfrac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}}$

$$\boxed{\begin{aligned}\text{Mean Curvature} = \tfrac{1}{2}\mathrm{tr}(S_p) &= \frac{eG - 2fF + Eg}{2(EG-F^2)} \\ &= \frac{f_{uu}(1+f_v^2) - 2f_{uv}f_uf_v + f_{vv}(1+f_u^2)}{2(1+f_u^2+f_v^2)^{3/2}}.\end{aligned}}$$

(b) Paraboloid: $f(u,v) = u^2+v^2$

$f_u = 2u \qquad f_v = 2v$

$f_{uu} = 2 \qquad f_{vv} = 2 \qquad f_{uv} = f_{vu} = 0.$

Gauss Curvature $K = \dfrac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$

$$= \frac{4}{(1+4u^2+4v^2)^2}$$

Mean Curvature: $H = \dfrac{f_{uu}(1+f_v^2) - 2f_{uv}f_uf_v + f_{vv}(1+f_u^2)}{2(1+f_u^2+f_v^2)^{3/2}}$

$$= \frac{2(1+4v^2) - 0 + 2(1+4u^2)}{2(1+4u^2+4v^2)^{3/2}}$$

$$= \frac{4+8v^2+8u^2}{2(1+4u^2+4v^2)^{3/2}}$$

Saddle: $f(u,v) = u^2 - v^2$

$f_u = 2u \qquad f_v = -2v$

$f_{uu} = 2 \qquad f_{vv} = -2 \qquad f_{uv} = f_{vu} = 0$

Gauss Curvature: $K = \dfrac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2} = \dfrac{-4}{(1+4u^2+4v^2)^2}$

Mean Curvature: $H = \dfrac{f_{uu}(1+f_v^2) - 2f_{uv}f_uf_v + f_{vv}(1+f_u^2)}{2(1+f_u^2+f_v^2)^{3/2}}$

$$= \frac{8v^2 - 8u^2}{2(1+4u^2+4v^2)^{3/2}}$$

[redacted]

(c) Recall that $\|(u,v)\| = u^2+v^2$

For the paraboloid: $K = \dfrac{4}{(1+4(u^2+v^2))^2} = \dfrac{4}{(1+4\|(u,v)\|)^2}$

$$H = \frac{4+8(u^2+v^2)}{2(1+4(u^2+v^2))^{3/2}} = \frac{4+8\|(u,v)\|}{2(1+4\|(u,v)\|)^{3/2}}$$

$\lim\limits_{\|(u,v)\|\to\infty} K = 0$ and $\lim\limits_{\|(u,v)\|\to\infty} = 0$.

For the saddle: $K = \dfrac{-4}{(1+4(u^2+v^2))^2} = \dfrac{-4}{(1+4\|(u,v)\|)^2}$

$$H = \frac{8(v^2-u^2)}{2(1+4(u^2+v^2))^{3/2}} = \frac{4(v^2-u^2)}{(1+4\|(u,v)\|)^{3/2}}$$

It's easy to see that $\lim\limits_{\|(u,v)\|\to\infty} K = 0$.

Now we consider the mean curvature. When $v \neq 0$, let $r = \frac{u}{v}$, so r indicates the direction of (u,v). We can write H as:

$$H = \frac{4v^2(1-r^2)}{(1+4\|(u,v)\|)^{3/2}} = \frac{4v^2(1-r^2)}{(1+4v^2(1+r^2))^{3/2}}$$

$\lim\limits_{\|(u,v)\|\to\infty} H$ depends on the value of r, i.e. It depends on the direction of (u,v), so this limit does not exist.