HW set 8 — Sebastian Seiveri

② Haberman 7.10.1 e) and g)

$$u_{tt} = c^2 \Delta u, \quad u(a,\theta,\varphi,t)=0$$

general solution to spherical wave equation:

$$u(r,\theta,\varphi,t) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{p=1}^{\infty} A_{nmp}\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\cos(m\theta)\,P_n^m(\cos\varphi)\cos(c\sqrt{\lambda}\,t)$$
$$+ \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{p=1}^{\infty} B_{nmp}\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\sin(m\theta)\,P_n^m(\cos\varphi)\cos(c\sqrt{\lambda}\,t)$$
$$+ \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{p=1}^{\infty} C_{nmp}\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\cos(m\theta)\,P_n^m(\cos\varphi)\sin(c\sqrt{\lambda}\,t)$$
$$+ \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{p=1}^{\infty} D_{nmp}\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\sin(m\theta)\,P_n^m(\cos\varphi)\sin(c\sqrt{\lambda}\,t)$$

e) $u(r,\theta,\varphi,0) = F(r,\varphi)\cos 3\theta, \quad u_t(r,\theta,\varphi,0)=0$

$u(r,\theta,\varphi,0) = F(r,\varphi)\cos 3\theta \Rightarrow B_{nmp}=0\ \forall n,m,p,\ A_{nmp}=0\ \forall n\forall p\forall m\neq 3$ and

$$F(r,\varphi) = \sum_{n=3}^{\infty}\sum_{p=1}^{\infty} A_{n3p}\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\,P_n^3(\cos\varphi)$$

$$\Rightarrow A_{n3p} = \frac{\int_0^{\pi}\int_0^{a} F(r,\varphi)\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\,P_n^3(\cos\varphi)\,r^2\sin\varphi\,dr\,d\varphi}{\int_0^{\pi}\int_0^{a}\frac{1}{r} J^2_{n+\frac12}(\sqrt{\lambda_{np}}\,r)(P_n^3)^2(\cos\varphi)\,r^2\sin\varphi\,dr\,d\varphi}$$

$u_t(r,\theta,\varphi,0)=0 \Rightarrow C_{nmp}, D_{nmp}=0\ \forall n,m,p$

g) $u(r,\theta,\varphi,0) = F(r), \quad u_t(r,\theta,\varphi,0)=0$

$u_t(r,\theta,\varphi,0)=0 \Rightarrow C_{nmp}, D_{nmp}=0\ \forall n,m,p$

$u(r,\theta,\varphi,0) = F(r) \Rightarrow B_{nmp}=0\ \forall n,m,p,\ A_{nmp}=0\ \forall n\neq 0\ \forall p\ \forall m\neq 0$ and — because F has no φ dependence

$$\cancel{F(r) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} A_{n0p}\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\,P_n^0(\cos\varphi)} \quad \Rightarrow n=0$$

$$\cancel{\Rightarrow A_{n0p} = \frac{\int_0^{\pi}\int_0^{a} F(r)\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{np}}\,r)\,P_n^0(\cos\varphi)\,r^2\sin\varphi\,dr\,d\varphi}{\int_0^{\pi}\int_0^{a}\frac{1}{r} J^2_{n+\frac12}(\sqrt{\lambda_{np}}\,r)(P_n^0)^2(\cos\varphi)\,r^2\sin\varphi\,dr\,d\varphi}}$$

$$\Rightarrow F(r) = \sum_{p=1}^{\infty} A_{00p}\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{0p}}\,r)$$

$$\Rightarrow A_{00p} = \frac{\int_0^{a} F(r)\frac{1}{\sqrt{r}} J_{n+\frac12}(\sqrt{\lambda_{0p}}\,r)\,r^2\,\cancel{\sin\varphi\,dr\,d\varphi}\,dr}{\int_0^{a}\frac{1}{r} J^2_{n+\frac12}(\sqrt{\lambda_{0p}}\,r)\,r^2\,dr}$$

② Haberman 8.2.1 (a), (b)

$u_t = k u_{xx} + Q(x)$, $u(x,0) = f(x)$

(a) $Q(x) = 0$, $u(0,t) = A$, $u_x(L,t) = B$

equilibrium sol'n u_E satisfies note: $u_E = u_E(x)$

$(u_E)_{xx} = 0 \Rightarrow (u_E)_x = c_0 \Rightarrow u_E = c_0 x + c_1$

BCs: $\left.\begin{array}{l} A = u_E(0) = c_1, \\ B = (u_E)_x(L) = c_0 \end{array}\right\} \Rightarrow u_E = Bx + A$

let $v(x,t) := u(x,t) - u_E(x)$
$= u(x,t) - Bx - A$

Then $v_t = u_t$, $v_{xx} = u_{xx}$ so $v_t = k v_{xx}$

and $v(0,t) = u(0,t) - A = 0$,
$v_x(L,t) = u_x(L,t) - B = 0$

and $v(x,0) = u(x,0) - Bx - A$
$= f(x) - Bx - A$

solve $v_t = k v_{xx}$, $v(0,t) = v_x(L,t) = 0$:

with $v(x,t) = X(x)\,T(t)$

$\Rightarrow v_t = X T'$, $v_{xx} = X'' T$

$\overset{PDE}{\Rightarrow} X T' = k X'' T$

$\Leftrightarrow \dfrac{T'}{kT} = \dfrac{X''}{X} = -\lambda \Leftrightarrow \begin{cases} T' + \lambda k T = 0 & (1) \\ X'' + \lambda X = 0, \ X(0) = 0, \ X'(L) = 0 & (2) \end{cases}$

(1) $T(t) = e^{-\lambda k t}$

(2) char. pol.: $r^2 = -\lambda \Rightarrow r = \pm i\sqrt{\lambda} \Rightarrow X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$\Rightarrow X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

BCs: $0 = X(0) = c_1$,

$0 = X'(L) = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) \overset{c_2,\lambda \neq 0}{\Rightarrow} 0 = \cos(\sqrt{\lambda}L)$

$\Rightarrow \sqrt{\lambda}L = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$

$\Rightarrow \lambda_n = \left(\dfrac{(n+\frac{1}{2})\pi}{L}\right)^2$ eigenvalues

So, eigenfunctions: $\sin\left(\dfrac{(n+\frac{1}{2})\pi}{L}x\right)$

general solution for v:

$$v(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(n+\frac{1}{2})\pi}{L} x\right) e^{-k\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 t}$$

general solution for u:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\Big(\overbrace{\frac{(n+\frac{1}{2})\pi}{L}}^{=\sqrt{\lambda_n}} x\Big) e^{-k\overbrace{\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2}^{=\lambda_n} t} + Bx + A$$

IC: $f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \sin(\sqrt{\lambda_n}x) + Bx + A$

$\Leftrightarrow$ $f(x) - Bx - A = \sum_{n=0}^{\infty} A_n \sin(\sqrt{\lambda_n} x)$

$\Rightarrow$ $A_n = \frac{2}{L}\int_0^L (f(x) - Bx - A)\sin(\sqrt{\lambda_n} x)\, dx \quad , n = 0,1,2,\ldots$

$\lim_{t\to\infty} u(x,t) = Bx + A = u_E$

(5) $Q(x) = 0$, $u_x(0,t) = 0$, $u_x(L,t) = B \neq 0$

equilibrium solution $u_E(x)$ satisfies:

$(u_E)_{xx} = 0 \Rightarrow u_E(x) = c_0 x + c_1$

BCs: $0 = (u_E)_x(0) = c_0$

$0 \neq B = (u_E)_x(L) = c_0 = 0$ ↯ So, no equilibrium solution exists

let $u_s(x) := \frac{Bx^2}{2L}$; then let $v(x,t) := u(x,t) - u_s(x)$;

we have $v_t = u_t$ and $v_{xx} = u_{xx} - \frac{B}{L}$ and

$v_x(0,t) = u_x(0,t) - (u_s)_x(x) = 0,$

$v_x(L,t) = u_x(L,t) - (u_s)_x(L) = B - B = 0$

$$\underset{v_t}{\overset{\shortparallel}{u_t}} = k\, \underset{v_{xx}+\frac{B}{L}}{\overset{\shortparallel}{u_{xx}}} \Rightarrow v_t = k v_{xx} + k\frac{B}{L}$$

so v satisfies the homogeneous BC problem with constant source:

$$v_t = k v_{xx} + k\frac{B}{L},$$
$$v_x(0,t) = v_x(L,t) = 0$$

⑨ Haberman 8.3.6.:

$$u_t = u_{xx} + e^{-2t}\sin 5x\,,\quad u(0,t)=1,\ u(\pi,t)=0,\ u(x,0)=0$$

Solution:

equilibrium solution satisfies: $u_E = u_E(x)$

$0 = (u_E)_{xx} \cancel{+ e^{-2t}\sin 5x}$ (note $e^{-2t}\sin 5x \xrightarrow{t\to\infty} 0$)

$\Rightarrow u_E(x) = c_0 x + c_1$

BC:

$$\left.\begin{aligned} 1 &= u_E(0) = c_1 \\ 0 &= u_E(\pi) = c_0\pi + 1 \Rightarrow c_0 = -\tfrac{1}{\pi} \end{aligned}\right\} \Rightarrow u_E(x) = 1 - \frac{x}{\pi}$$

Now, consider $v(x,t) := u(x,t) - u_E(x)$:

$v_t = u_t$, $v_{xx} = u_{xx}$, so v should satisfy $v_t = v_{xx} + e^{-2t}\sin 5x$;

$v(0,t) = u(0,t) - u_E(0) = 1 - 1 = 0$,

$v(\pi,t) = u(\pi,t) - u_E(\pi) = 0 - 1 + 1 = 0$

IC: $v(x,0) = u(x,0) - u_E(x) = \frac{x}{\pi} - 1$

Solve the following problem for v: $v_t = v_{xx} + e^{-2t}\sin 5x$,
$v(0,t) = v(\pi,t) = 0$, $v(x,0) = \frac{x}{\pi} - 1$

write $v(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(\overset{\frac{n\pi x}{\pi}}{\overset{\shortparallel}{nx}})$ (eigenfunction expansion)

IC: $\frac{x}{\pi} - 1 = v(x,0) = \sum_{n=1}^{\infty} B_n(0)\sin(nx)$

$\Rightarrow B_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{x}{\pi} - 1\right)\sin(nx)\,dx = -\frac{2}{\pi n}\quad \forall n$

$v_t = \sum_{n=1}^{\infty} B_n'(t)\sin(nx)$, $v_{xx} = \sum_{n=1}^{\infty} -n^2 B_n(t)\sin(nx)$

$\overset{\text{PDE}}{\Rightarrow} \sum_{n=1}^{\infty} B_n'(t)\sin(nx) = \sum_{n=1}^{\infty} -n^2 B_n(t)\sin(nx) + e^{-2t}\sin(5x)$

$\Rightarrow$ get the following ODEs:

$$\begin{cases} B_n'(t) = -n^2 B_n(t) \quad \forall n \neq 5 & (1) \\ B_5'(t) = -25 B_5(t) + e^{-2t}\,,\ n = 5 & (2) \end{cases}$$

(1) $B_n(t) = B_n(0)e^{-n^2 t}$

(2) $B_5'(t) + 25 B_5(t) = e^{-2t}$ integrating factor: $e^{\int 25\,dt} = e^{25t}$

$\Rightarrow \underbrace{B_5'(t)e^{25t} + 25e^{25t}B_5(t)}_{=(B_5(t)e^{25t})'} = e^{23t}$

$\Rightarrow B_5(t)e^{25t} = \int e^{23t}\,dt = \frac{1}{23}e^{23t} + \overset{\text{const.}}{\overset{\shortparallel}{c}}$

$$\Rightarrow \quad B_5(t) = \frac{1}{23} e^{-2t} + c\, e^{-25t}$$

at $t=0$: $\quad B_5(0) = \frac{1}{23} + c \quad \Rightarrow \quad c = B_5(0) - \frac{1}{23}$,

$$\text{so} \quad B_5(t) = \frac{1}{23} e^{-2t} + \left(B_5(0) - \frac{1}{23}\right) e^{-25t}$$

So, $$v(x,t) = \sum_{\substack{n=1 \\ n\neq 5}}^{\infty} B_n(0) e^{-n^2 t} \sin(nx) + \left(\frac{1}{23} e^{-2t} + \left(B_5(0) - \frac{1}{23}\right) e^{-25t}\right) \sin(5x)$$

Then, $u(x,t) = v(x,t) + 1 - \frac{x}{\pi}$ gives

$$u(x,t) = 1 - \frac{x}{\pi} - \sum_{\substack{n=1 \\ n\neq 5}}^{\infty} \frac{2}{\pi n} e^{-n^2 t} \sin(nx) + \left(\frac{1}{23} e^{-2t} - \left(\frac{2}{5\pi} + \frac{1}{23}\right) e^{-25t}\right) \sin(5x)$$

$$= 1 - \frac{x}{\pi} + \frac{1}{23}\left(e^{-2t} - e^{-25t}\right) \sin(5x) - \sum_{n=1}^{\infty} \frac{2}{\pi n} e^{-n^2 t} \sin(nx)$$