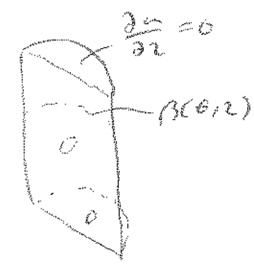
HW 7

(7.) Haberman 7.9.2 (a), (b)

(b) $u(r,\theta,0)=0,\ \frac{\partial u}{\partial z}(r,\theta,H)=0,\ u(r,0,z)=0,$
$u(r,\pi,z)=0,\ u(a,\theta,z)=\beta(\theta,z)$



$$\Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}$$

$$\Rightarrow\begin{cases} Z''=\lambda Z, & Z(0)=0,\ Z'(H)=0 \quad (1)\\ T''+\mu T=0, & T(0)=T(\pi)=0 \quad (2)\\ r(rR')'+(\lambda r^2-\mu)R=0, & \cancel{R(a)=\beta(\theta,z)} \quad (3)\end{cases}$$

$Z''-\lambda Z=0$
$\Rightarrow \int_0^H ZZ''-\int_0^H\lambda Z^2=0$
$\Rightarrow \underbrace{ZZ'\big|_0^H}_{=0}-\int_0^H(Z')^2-\lambda\int_0^H Z^2=0$
$\Leftrightarrow \lambda=-\frac{\int_0^H(Z')^2\,dz}{\int_0^H Z^2\,dz}<0$

(2) $\mu=n^2,\ T_n(\theta)=\sin(n\theta),\ n=1,2,\ldots$

(1) $Z''=\lambda Z \Rightarrow r^2=\lambda \overset{\lambda<0}{\Rightarrow} r=\pm i\sqrt{-\lambda} \Rightarrow Z=c_1\cos(\sqrt{-\lambda}z)+c_2\sin(\sqrt{-\lambda}z)$

BCs: $0=Z(0)=c_1$

$0=Z'(H)=c_2\sqrt{-\lambda}\cos(\sqrt{-\lambda}H) \Rightarrow \sqrt{-\lambda}H=(\tfrac12+m)\pi$

$\Rightarrow \lambda_m=-\left(\frac{(\frac12+m)\pi}{H}\right)^2,\ m=0,1,2,\ldots$

$Z_m(z)=\sin(\sqrt{-\lambda_m}z)$

(3) $r(rR')'+\left(-\left((\tfrac12+m)\tfrac{\pi}{H}\right)^2r^2-n^2\right)R=0$

$R(r)=c_1I_n\left((\tfrac12+m)\tfrac{\pi}{H}r\right)+c_2K_n(\ldots)$

gen. sol'n:

$$u(r,\theta,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{mn}I_n\left((\tfrac12+m)\tfrac{\pi}{H}r\right)\sin(n\theta)\sin\left((\tfrac12+m)\tfrac{\pi}{H}z\right),$$

$$A_{mn}=\frac{\int_0^\pi\int_0^H\beta(\theta,z)\sin(n\theta)\sin\left((\frac12+m)\frac{\pi}{H}z\right)d\theta\,dz}{I_n\left((\frac12+m)\frac{\pi}{H}a\right)\int_0^\pi\sin^2(n\theta)\,d\theta\int_0^H\sin^2\left((\frac12+m)\frac{\pi}{H}z\right)dz}$$

(a) $u(r,\theta,0)=0,\ u(r,\theta,H)=\alpha(r,\theta),\ u(r,0,z)=0,$
$u(r,\pi,z)=0,\ u(a,\theta,z)=0$ ← gives coefficients

$u(r,\theta,z)=R(r)T(\theta)Z(z)$
$u(r,\theta,0)=R(r)T(\theta)Z(0)=0$
$\Rightarrow Z(0)=0$
$u(r,\theta,H)=R(r)T(\theta)Z(H)=\alpha(r,\theta)$

$$\Rightarrow\begin{cases} Z''=\lambda Z, & Z(0)=0,\ \cancel{Z(H)=\alpha(r,\theta)} \quad (1)\\ T''+\mu T=0, & T(0)=T(\pi)=0 \quad (2)\\ r(rR')'+(\lambda r^2-\mu)R=0, & R(a)=0 \quad (3)\end{cases}$$

(2) $\mu=n^2,\ T_n(\theta)=\sin(n\theta),\ n=1,2,\ldots$

(3) $r(rR')' + (\lambda r^2 - n^2)R = 0$

$\Rightarrow R(r) = c_1 J_n(\sqrt{\lambda} r) + \cancel{c_2 Y_n(\sqrt{\lambda} r)}$

b/c $|R(0)| < \infty$

BC: $0 = R(a) = c_1 J_n(\sqrt{\lambda} a)$

$\Rightarrow \sqrt{\lambda_{nm}}\, a = z_{nm}$ (mth zero of nth Bessel function), $m = 1, 2, \ldots$

$\Rightarrow \lambda_{nm} = \left(\frac{z_{nm}}{a}\right)^2$

(1) $Z'' = \lambda Z$ (no eigenvalue problem) $\lambda \geq 0$

$(\Rightarrow)$ $Z'' - \lambda Z = 0$

$\Rightarrow$ ch. pol.: $r^2 = \lambda$ $\Rightarrow$ $r = \pm\sqrt{\lambda}$ $\Rightarrow$ $Z = c_1 e^{\sqrt{\lambda} z} + c_2 e^{-\sqrt{\lambda} z}$

$= \tilde{c}_1 \cosh(\sqrt{\lambda} z) + \tilde{c}_2 \sinh(\sqrt{\lambda} z)$

$0 = Z(0) = \tilde{c}_1$

$\Rightarrow Z = \tilde{c}_2 \sinh(\sqrt{\lambda} z)$

So gen. sol'n:

$$u(r,\theta,z) = \sum_{n,m=1}^{\infty} A_{nm} J_n(\sqrt{\lambda_{nm}}\, r) \sin(n\theta) \sinh(\sqrt{\lambda} z),$$

$$A_{nm} = \frac{\int_0^a \int_0^\pi \alpha(r,\theta) J_n(\sqrt{\lambda} r) r \sin(n\theta)\, d\theta\, dr}{\int_0^a J_n^2(\sqrt{\lambda} r) r\, dr \int_0^\pi \sin^2(n\theta)\, d\theta \sinh(\sqrt{\lambda} H)}$$

① Haberman 7.7.1.:

$u_{tt} = c^2 \Delta u \quad , \; u(a,\theta,t) = 0, \; u(r,\theta,0) = 0, \; u_t(r,\theta,0) = \alpha(r)\sin 3\theta$

$u = u(r,\theta,t) = \varphi(r,\theta)\, h(t)$

$\Rightarrow u_{tt} = \varphi h'' , \; \Delta u = \Delta\varphi\, h$

PDE $\Rightarrow \varphi h'' = c^2 \Delta\varphi \cdot h \iff \dfrac{h''}{c^2 h} = \dfrac{\Delta\varphi}{\varphi} = -\lambda$

$\iff \begin{cases} h'' + c^2\lambda h = 0 & (1) \\ \Delta\varphi + \lambda\varphi = 0 & (2) \end{cases}$

(1) $r^2 = -c^2\lambda \Rightarrow r = \pm i\sqrt{\lambda}\, c \Rightarrow h(t) = c_1 \cos(\sqrt{\lambda}\, c t) + c_2 \sin(\sqrt{\lambda}\, c t)$

(2) $\Delta\varphi + \lambda\varphi = 0 \quad , \varphi = R(r)\,T(\theta)$

$\iff \frac{1}{r}(rR')'T + \frac{1}{r^2} R T'' + \lambda R T = 0$

$\iff \frac{r}{R}(rR')' + \lambda r^2 = -\frac{T''}{T} = \mu \iff \begin{cases} T'' + \mu T = 0 & , \mu = n^2, \; T(\theta) = \cos(n\theta), \sin(n\theta) \quad n = 0,1,2,\dots \\ r(rR')' + (\lambda r^2 - \mu)R = 0 \quad (3) & , R(a) = 0 \text{ from } u(a,\theta,t) = 0 \end{cases}$

(3) $R(r) = c_1 J_n(\sqrt{\lambda}\, r) + \cancel{c_2 Y_n(\sqrt{\lambda}\, r)}$ b/c $|R(0)| < \infty$

$0 = R(a) = c_1 J_n(\sqrt{\lambda}\, a)$

$\overset{c_1 \neq 0}{\Rightarrow} J_n(\sqrt{\lambda}\, a) = 0$

$\Rightarrow \lambda = \left(\frac{z_{nm}}{a}\right)^2$, z_{nm} mth zero of nth bessel function, $m = 1,2,\dots$

gen. sol'n:

$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} J_n(\sqrt{\lambda}\, r)\cos(n\theta)\cos(\sqrt{\lambda}\, ct)$

$+ \sum_n \sum_m B_{nm} J_n(\sqrt{\lambda}\, r)\sin(n\theta)\cos(\sqrt{\lambda}\, ct)$

$+ \sum_n \sum_m C_{nm} J_n(\sqrt{\lambda}\, r)\cos(n\theta)\sin(\sqrt{\lambda}\, ct)$

$+ \sum_n \sum_m D_{nm} J_n(\sqrt{\lambda}\, r)\sin(n\theta)\sin(\sqrt{\lambda}\, ct)$

ICs: $0 = u(r,\theta,0) \Rightarrow A_{nm}, B_{nm} = 0$

$\alpha(r)\sin 3\theta = u_t(r,\theta,0) = \sum_n\sum_m C_{nm} J_n(\sqrt{\lambda}\, r)\cos(n\theta)\sqrt{\lambda}\, c$

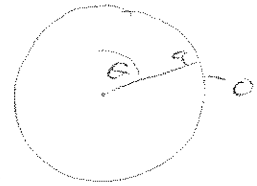
$+ \sum_n\sum_m D_{nm} J_n(\sqrt{\lambda}\, r)\sin(n\theta)\sqrt{\lambda}\, c$

$\Rightarrow C_{n,m} = 0 \; , D_{nm} = 0$ but $D_{3m} \neq 0$

$D_{3m} = \dfrac{\int_0^a \cancel{\int_{-\pi}^{\pi}} \alpha(r)\cancel{\sin(3\theta)} J_n(\sqrt{\lambda_{3m}}\, r)\, r\, \cancel{\sin(3\theta)\, d\theta}\, dr}{\int_0^a J_n^2(\sqrt{\lambda_{3m}}\, r)\, r\, dr \; \cancel{\int_{-\pi}^{\pi}\sin^2(3\theta)\, d\theta}\; \sqrt{\lambda}\, c}$

$\boxed{\text{so } u(r,\theta,t) = \sum_{m=1}^{\infty} D_{3m} J_3(\sqrt{\lambda_{3m}}\, r)\sin(3\theta) \cdot \sin(\sqrt{\lambda_{3m}}\, ct)}$

~~7.7.7~~ (3.) Haberman 7.7.7



$u_t = k\Delta u$, $u(r,\theta,0) = f(r,\theta)$

$u = \varphi h$

$\Rightarrow \varphi h' = k\Delta\varphi h$

$\Rightarrow \frac{h'}{kh} = \frac{\Delta\varphi}{\varphi} = -\lambda \Rightarrow \begin{cases} h' + \lambda k h = 0 \\ \Delta\varphi + \lambda\varphi = 0 \end{cases}$ (2) $\Rightarrow h(t) = e^{-\lambda k t}$ (1)

(2) $\Delta\varphi + \lambda\varphi = 0$, $\varphi = RT$

$T(\theta) = \cos(n\theta), \sin(n\theta)$, $\mu = n^2$, $n = 0,1,2,\ldots$

$R(r) = J_n(\sqrt{\lambda}\, r)$, $\lambda_{nm} = \left(\frac{z_{nm}}{a}\right)^2$, $m = 1,2,\ldots$ $\quad \lambda \geq 0\ \forall n,m$

So, gen. solution:

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}\, r)\cos(n\theta)\, e^{-\lambda k t} A_{nm}$$
$$+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}\, r)\sin(n\theta)\, e^{-\lambda k t} B_{nm}$$

IC: $f(r,\theta) = u(r,\theta,0) = \sum_n \sum_m A_{nm} J_n(\sqrt{\lambda}\, r)\cos(n\theta) + \sum_n \sum_m J_n(\sqrt{\lambda}\, r)\sin(n\theta) B_{nm}$

$$\Rightarrow \quad A_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\cos(n\theta)\, J_n(\sqrt{\lambda}\, r)\, r\, dr\, d\theta}{\int_0^a J_n^2(\sqrt{\lambda}\, r)\, r\, dr \int_{-\pi}^{\pi}\cos^2(n\theta)\, d\theta},$$

B_{nm} = same w/ $\sin(n\theta)$ instead of $\cos(n\theta)$

$\lim_{t\to\infty} u(r,\theta,t) = 0$