HW 10

1. $3u_x + 5u_t = 0$, IC: $u(x,0) = f(x)$

Fourier transform the PDE: $u = u(x,t)$, $\hat{u} = \hat{u}(\xi, t)$

$0 = \widehat{3u_x + 5u_t} = 3\widehat{u_\xi} + 5\hat{u}_t = 3i\xi\hat{u} + 5\hat{u}_t$

$\Leftrightarrow \quad \hat{u}_t + \frac{3}{5}i\xi\hat{u} = 0$ is an ODE in t with

solution $\hat{u}(\xi,t) = c(\xi)e^{-\frac{3}{5}i\xi t}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi,0) = \hat{f}(\xi)$, so

$\hat{f}(\xi) = \hat{u}(\xi,0) = c(\xi)$

it follows that $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-\frac{3}{5}i\xi t}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\frac{3}{5}i\xi t}e^{i\xi x}\,d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x-\frac{3}{5}t)}\,d\xi = f(x - \tfrac{3}{5}t)$$

So, $u(x,t) = f(x - \frac{3}{5}t)$.

2. $3t u_x + 5u_t = 0$, IC: $u(x,0) = f(x)$

Fourier transform the PDE:

$0 = \widehat{3t u_x + 5u_t} = 3t\widehat{u_\xi} + 5\hat{u}_t = 3ti\xi\hat{u} + 5\hat{u}_t$

$\Leftrightarrow \quad \hat{u}_t + \frac{3}{5}i\xi t\hat{u} = 0$, integrating factor: $e^{\int \frac{3}{5}i\xi t\,dt} = e^{\frac{3}{10}i\xi t^2}$

$\Leftrightarrow \quad \underbrace{\hat{u}_t e^{\frac{3}{10}i\xi t^2} + \frac{3}{5}i\xi t\hat{u}e^{\frac{3}{10}i\xi t^2}}_{= \left(\hat{u}e^{\frac{3}{10}i\xi t^2}\right)_t} = 0$

$\Rightarrow \quad \hat{u}e^{\frac{3}{10}i\xi t^2} = c(\xi)$

$\Leftrightarrow \quad \hat{u}(\xi,t) = c(\xi)e^{-\frac{3}{10}i\xi t^2}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi,0) = \hat{f}(\xi)$, so $\hat{f}(\xi) = \hat{u}(\xi,0) = c(\xi)$

it follows that $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-\frac{3}{10}i\xi t^2}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\frac{3}{10}i\xi t^2}e^{i\xi x}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x-\frac{3}{10}t^2)} = f(x - \tfrac{3}{10}t^2)$$

So, $u(x,t) = f(x - \frac{3}{10}t^2)$.

(3.) $u_t + u_x + u = 0$, IC: $u(x,0) = f(x)$

Fourier transform the PDE:

$$0 = \widehat{u_t + u_x + u} = \hat{u}_t + \widehat{u_x} + \hat{u} = \hat{u}_t + i\xi\hat{u} + \hat{u} = \hat{u}_t + (i\xi + 1)\hat{u}$$ is an ODE in t

with solution $\hat{u}(\xi, t) = c(\xi)e^{-(i\xi+1)t}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi, 0) = \hat{f}(\xi)$,

so $\hat{f}(\xi) = \hat{u}(\xi, 0) = c(\xi)$

it follows that $\hat{u}(\xi, t) = \hat{f}(\xi)e^{-(i\xi+1)t}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-(i\xi+1)t}e^{i\xi x}\,d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x-t)}\,d\xi\, e^{-t} = e^{-t}f(x-t)$$

So, $u(x,t) = e^{-t}f(x-t)$.

This solve the PDE:

$$u_t = +e^{-t}\cdot f'(x-t)(-1) = \cancel{e^{-t}f'(x-t)} + (-e^{-t})f(x-t)$$
$$u_x = e^{-t}f'(x-t)$$
$$u = e^{-t}f(x-t)$$
$$\Rightarrow u_t + u_x + u = -e^{-t}f'(x-t) - e^{-t}f(x-t) + e^{-t}f'(x-t) + e^{-t}f(x-t) = 0 \checkmark$$

(5.) $u_t = ku_{xx} + u$, $u(x,0) = f(x)$, $-\infty < x < \infty$, $t > 0$ (see also lecture 21)

Fourier transform the PDE:

$$\widehat{u_t} = \widehat{ku_{xx} + u} = k\widehat{u_{\xi\xi}} + \hat{u} = -k\xi^2\hat{u} + \hat{u} = (1 - k\xi^2)\hat{u}$$ is an ODE in t

with solution $\hat{u}(\xi, t) = c(\xi)e^{(1-k\xi^2)t}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi, 0) = \hat{f}(\xi)$, so $\hat{f}(\xi) = \hat{u}(\xi, 0) = c(\xi)$

it follows that $\hat{u}(\xi, t) = \hat{f}(\xi)e^{(1-k\xi^2)t}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{(1-k\xi^2)t}e^{i\xi x}\,d\xi$$

What g satisfies $\hat{g}(\xi) = e^{(1-k\xi^2)t}$?

$$\hat{g}(\xi) = e^{(1-k\xi^2)t} \Rightarrow g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\xi)e^{i\xi x}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(1-k\xi^2)t + i\xi x}\,d\xi$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k\xi^2 t + i\xi x}\,d\xi\right)}_{= \frac{1}{\sqrt{2kt}}e^{-\frac{x^2}{4kt}} \text{ (see lecture 21)}} e^{t}$$

$$= \frac{1}{\sqrt{2kt}} e^{t - \frac{x^2}{4kt}}$$

Now, $\hat{u}(\xi, t) = \hat{f}(\xi)\, \hat{g}(\xi)$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} (f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2kt}} e^{t - \frac{(x-y)^2}{4kt}} dy$$

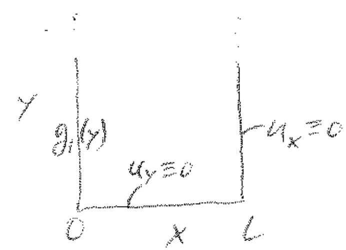
$$= \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} f(y) e^{t - \frac{(x-y)^2}{4kt}} dy$$

$$= \frac{1}{\sqrt{4\pi k t}} e^{t} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4kt}} dy$$

7. Haberman 10.6.2. (a), (b)

(b) $u_{xx} + u_{yy} = 0$, $x \in (0, L)$, $y > 0$

$u(0,y) = g_1(y)$, $u_x(L,y) = 0$, $u_y(x,0) = 0$



let ^ denote Fourier cosine transform

we choose Fourier cosine transform b/c of $u_y(x,0) = 0$ (Neumann)

transform y variable ($y \to \xi$)

$u = u(x,y)$,
$\hat{u} = \hat{u}(x, \xi)$

$$0 = \hat{0} = \widehat{u_{xx} + u_{yy}} = \hat{u}_{xx} - \frac{2}{\pi} \underbrace{u_y(x,0)}_{=0} - \xi^2 \hat{u} = \hat{u}_{xx} - \xi^2 \hat{u} \text{ is an ODE in } x$$

with solution $\hat{u}(x, \xi) = \tilde{c}_1(\xi) e^{\xi x} + \tilde{c}_2(\xi) e^{-\xi x}$

change of basis $= c_1(\xi) \cosh(\xi x) + c_2(\xi) \cosh(\xi(L-x))$

apply BCs

- $u(0,y) = g_1(y) \Rightarrow \hat{u}(0,\xi) = \hat{g}_1(\xi)$, so $\hat{g}_1(\xi) = \hat{u}(0,\xi) = c_1(\xi) + c_2(\xi) \cosh(\xi L)$
- $u_x(L,y) = 0 \Rightarrow \hat{u}_x(L,\xi) = 0$, so $0 = \hat{u}_x(L,\xi) = c_1(\xi)\,\xi \sinh(L\xi) + c_2(\xi)\,\xi(-1)\underbrace{\sinh(0)}_{=0}$

$$= c_1(\xi)\, \xi \sinh(L\xi)$$

$$\Rightarrow c_1(\xi) \equiv 0$$

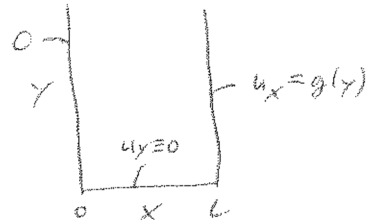
$$\Rightarrow c_2(\xi) = \frac{\hat{g}_1(\xi)}{\cosh(\xi L)}$$

So, $\hat{u}(x, \xi) = \dfrac{\hat{g}_1(\xi)}{\cosh(\xi L)} \cosh(\xi(L-x))$.

(a) $u_{xx} + u_{yy} = 0 \;,\; x \in (0,L) \;,\; y > 0$

$u_x(0,y) = 0 \;,\; u_x(L,y) = g(y) \;,$

$u_y(x,0) = 0$



$\hat{}$ denotes Fourier cosine transform

$y \longrightarrow \xi$

$$0 = \hat{0} = \widehat{u_{xx} + u_{yy}}$$

$$= \hat{u}_{xx} + \widehat{u_{\xi\xi}} = \hat{u}_{xx} - \frac{2}{\pi}\underbrace{u_y(x,0)}_{\equiv 0} - \xi^2 \hat{u}(x,\xi) = \hat{u}_{xx} - \xi^2 \hat{u} \quad \text{is an ODE in } x$$

with solution

$$\hat{u}(x,\xi) = \tilde{c}_1(\xi)\, e^{\xi x} + \tilde{c}_2(\xi)\, e^{-\xi x}$$

change of basis

$$= c_1(\xi)\cosh(\xi x) + c_2(\xi)\cosh(\xi(L-x))$$

apply BC: $u_x(0,y) = 0 \Rightarrow \hat{u}_x(0,\xi) = 0$, so

$$0 = \hat{u}_x(0,\xi) = c_1(\xi)\,\xi\,\underbrace{\sinh(0)}_{=0} + c_2(\xi)(-\xi)\sinh(\xi L)$$

$$= -c_2(\xi)\,\xi\sinh(\xi L)$$

$$\Rightarrow \quad c_2(\xi) \equiv 0$$

$u_x(L,y) = g(y) \Rightarrow \hat{u}_x(L,\xi) = \hat{g}(\xi)$, so

$$\hat{g}(\xi) = \hat{u}_x(L,\xi) = c_1(\xi)\,\xi\sinh(\xi L)$$

$$\Rightarrow \quad c_1(\xi) = \frac{\hat{g}(\xi)}{\xi\sinh(\xi L)}$$

So, $\hat{u}(x,\xi) = \dfrac{\hat{g}(\xi)}{\xi\sinh(\xi L)}\cosh(\xi x)$.