Section 3.3 Derivative Formulas

Let \( f(x) \) and \( g(x) \) be differentiable functions and let \( k \) be a constant.

1. Constant function

\[
\begin{align*}
y &= k \\
y' &= 0
\end{align*}
\]

2. Linear function

\[
\begin{align*}
y &= mx + b \\
y' &= m
\end{align*}
\]

3. Constant Multiple Rule

\[
\begin{align*}
y &= kf(x) \\
y' &= kf'(x)
\end{align*}
\]
(from last class)

4. The Power Rule

\[
\begin{align*}
y &= x^n, \quad n \text{ any real number} \\
y' &= nx^{n-1}
\end{align*}
\]

\[
\begin{align*}
ex : y &= x^4 \\
y' &= 4x^3
\end{align*}
\]

5. The Sum Rule

\[
\begin{align*}
y &= f(x) + g(x) \\
y' &= f'(x) + g'(x)
\end{align*}
\]

6. The Difference Rule

\[
\begin{align*}
y &= f(x) - g(x) \\
y' &= f'(x) - g'(x)
\end{align*}
\]
(from last class)

\[
\begin{align*}
ex : y &= 4x - 7x^2 \\
y' &= 4 - 14x
\end{align*}
\]

7. The Product Rule

\[
\begin{align*}
y &= f(x)g(x) \\
y' &= f'(x)g(x) + f(x)g'(x)
\end{align*}
\]

\[
\begin{align*}
(\text{1st})' (\text{2nd}) + (\text{1st})(\text{2nd})'
\end{align*}
\]

\[
\begin{align*}
ex : \text{Let } f(x) = \sqrt{x} \cdot g(x), \quad g(4) = 8, \quad \text{and } g'(4) = 7. \text{ Find } f'(4).
\end{align*}
\]

\[
\begin{align*}
f'(x) &= \frac{1}{2} x^{-1/2} \cdot g(x) + \sqrt{x} g'(x) = \frac{1}{2\sqrt{x}} \cdot g(x) + \sqrt{x} g'(x) \\
f'(4) &= \frac{1}{2\sqrt{4}} \cdot g(4) + \sqrt{4} g'(4) = \frac{1}{4} \cdot 8 + 2 \cdot 7 = 16
\end{align*}
\]
8. The Quotient Rule

\[ y = \frac{f(x)}{g(x)} \quad \Rightarrow \quad y' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \]

\[ y' = \frac{(top')(bottom) - (top)(bottom')}{[bottom]^2} \]

ex: Let \( y = \frac{3x+1}{x^2 + 1} \). Find \( y'(1) \).

\[ y'(x) = \frac{3(x^2+1) - (3x+1)(2x)}{(x^2+1)^2} \]

\[ y'(1) = \frac{3(1+1) - (3+1)(2)}{(1+1)^2} = \frac{6 - 8}{4} = \frac{-2}{4} = -\frac{1}{2} \]

ex: Find the equation of the tangent line to \( y = \frac{3x+1}{x^2 + 1} \) at \((1, 2)\).

\[ m = y'(1) \quad y = mx + b \quad x = 1, \ y = 2, \ m = -\frac{1}{2}, \ b = ? \]

\[ y'(1) = -\frac{1}{2} \quad 2 = -\frac{1}{2}(1) + b \]

\[ 2 + \frac{1}{2} = b \quad \Rightarrow \ b = \frac{5}{2} \quad y = -\frac{1}{2}x + \frac{5}{2} \]

The normal line to a curve at a point \( P \) is the line through \( P \) that is perpendicular to the tangent line at \( P \).

ex: Find the equation of the normal line to \( y = \frac{3x+1}{x^2 + 1} \) at \((1, 2)\).

\[ m_{\text{tangent}} = -\frac{1}{2} \quad y = mx + b \quad x = 1, \ y = 2, \ m = 2, \ b = ? \]

\[ m_{\text{normal}} = 2 \quad 2 = 2(1) + b \quad \Rightarrow \ b = 0 \quad y = 2x \]
3.3 Derivative Formulas

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ex: Let \( Q(x) = \frac{F(x)}{G(x)} \) where \( F \) and \( G \) are the functions whose graphs are shown. Find \( Q'(7) \).

\[
Q'(x) = \frac{F'(x)G(x) - F(x)G'(x)}{[G(x)]^2}
\]

\[
Q'(7) = \frac{F'(7)G(7) - F(7)G'(7)}{[G(7)]^2}
\]

\[
F(7) = 5 \quad F'(7) = \frac{1}{4}
\]

\[
G(7) = 1 \quad G'(7) = \frac{-2}{3}
\]

\[
Q'(7) = \frac{\frac{1}{4} (1) - 5 \left( \frac{-2}{3} \right)}{1} = \frac{\frac{10}{3} + \frac{3}{40}}{12} = \frac{43}{12}
\]

ex: Find the points on the curve \( y = 2x^3 + 3x^2 - 12x + 1 \) where the tangent line is horizontal.

\[
y = 2x^3 + 3x^2 - 12x + 1
\]

\[
y' = 6x^2 + 6x - 12 = 0
\]

set

\[
solve for x
\]

\[
6(x^2 + x - 2) = 0
\]

\[
6(x + 2)(x - 1) = 0
\]

either \( x + 2 = 0 \) or \( x - 1 = 0 \)

\[
x = -2 \quad \text{or} \quad x = 1
\]

\[
y(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 2(-8) + 3(4) + 24 + 1 = -16 + 12 + 25 = 1\]

\[
y(1) = 2(1) + 3(1) - 12(1) + 1 = 2 + 3 - 12 + 1 = -6
\]

\[
y(-2) = 21 \quad y(1) = -6
\]

\[
(-2, 21) \quad (1, -6)
\]
ex: Find the equation of the tangent lines to the curve

\[ y = \frac{x-1}{x+1} \]

that are parallel to the line \( x - 2y = 2 \).

\[ y' = \frac{1}{2} \]

cross multiply

\[ x + 1 = 2 \Rightarrow x = 1 \]

\( y(1) = \frac{1-1}{1+1} = 0 \)

(1,0) pt. of tangency

\[ m = \frac{1}{2} \]

\[ y = mx + b \]

\[ 0 = \frac{1}{2} + b \Rightarrow b = -\frac{1}{2} \]

\[ y = \frac{1}{2} x - \frac{1}{2} \]

\[ x + 1 = -2 \Rightarrow x = -3 \]

\( y(-3) = \frac{-3-1}{-3+1} = \frac{3}{2} \)

\( (-3,2) \) pt. of tangency

\[ m = \frac{1}{2} \]

\[ y = mx + b \]

\[ 2 = \frac{1}{2} (-3) + b \Rightarrow b = \frac{7}{2} \]

\[ y = \frac{1}{2} x + \frac{7}{2} \]

\[ \Rightarrow m_{tangent} = \frac{1}{2} \]

\[ \Rightarrow x + 1 = \pm 2 \]

ex: Find the value of \( c \) such that the line \( y = \frac{3}{2} x + 6 \) is tangent to the curve \( y = c \sqrt{x} \).

We have 2 unknowns: \( c \) and the \( x \)-value of the point of tangency.

call this \( x_0 \)

At the point of tangency, the curve and the tangent line have the same \( y \)-value.

\[ c \sqrt{x_0} = \frac{3}{2} x_0 + 6 \]

At the point of tangency, the slope of the tangent line is \( \frac{3}{2} \). \( \Rightarrow y'(x_0) = \frac{3}{2} \)

\[ y' = c \frac{1}{2} x^{-1/2} \]

\[ y' = c \frac{1}{2} \sqrt{x} \]

\[ \Rightarrow \frac{c}{2 \sqrt{x_0}} = \frac{3}{2} \]

\[ 3 \sqrt{x_0} \sqrt{x_0} = \frac{3}{2} x_0 + 6 \]

\[ 3 x_0 - \frac{3}{2} x_0 = 6 \]

\[ \Rightarrow \frac{3}{2} x_0 = 6 \]

\[ x_0 = 4 \]

plug this into the other equation

\[ \Rightarrow c = 3 \sqrt{4} \]

\[ c = 6 \]
ex: Let
\[
  f(x) = \begin{cases} 
    2 - x & \text{if } x \leq 1 \\
    x^2 - 2x + 2 & \text{if } x > 1 
  \end{cases}
\]
Is \( f \) differentiable at 1? Sketch the graphs of \( f \) and \( f' \).

In order for \( f \) to be differentiable at 1 it must be continuous at 1.

Is \( f \) continuous at 1? If the answer is no, then we are done.

\[
  \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2 - x) = 1 \\
  \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 2x + 2) = 1
\]

Yes \( f \) continuous at 1.

In order for \( f \) to be differentiable, the slope of the tangent line from the left must equal the slope of the tangent line from the right.

\[
  f'(x) = \begin{cases} 
    -1 & \text{if } x \leq 1 \\
    2x - 2 & \text{if } x > 1 
  \end{cases}
\]

\[
  \lim_{x \to 1^-} f'(x) = -1 \\
  \lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} (2x - 2) = 0
\]

\( f \) is not differentiable at 1.