3.7 Implicit Differentiation

\[
y = (4x - x^2)^5 \quad y = x \sin \sqrt{x} \quad y = \sqrt{x^2 + 1} \quad y = \tan^2 (3x^{2/3})
\]

All functions where \( y \) is defined **explicitly** in terms of \( x \).

\[
x^2 y^3 - 2xy = 6x + y + 19
\]

\( y \) is defined **implicitly** in terms of \( x \)

There is no way to solve for \( y \)

\( y = f'(x) \) but you won't be able to find a formula for it

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Find the slope of the tangent line to the curve

\[
x^2 y^3 - 2xy = 6x + y + 19 \text{ at the point } (-2,1).
\]

To reinforce how the method works, we sub in \( f(x) \) for \( y \)

\[
x^2 \left[ f(x) \right]^3 - 2x \left[ f(x) \right] = 6x + f(x) + 19
\]

\[
2x \left[ f(x) \right]^3 + 3x^2 \left[ f(x) \right]^2 f'(x) - 2 \left[ f(x) \right] - 2xf'(x) = 6 + f'(x)
\]

Put all terms with \( f'(x) \) on the left side

\[
3x^2 \left[ f(x) \right]^2 f'(x) - 2xf'(x) - f'(x) = 6 - 2x \left[ f(x) \right]^3 + 2 \left[ f(x) \right]
\]
3x^2\left[f(x)\right]^2 f'(x) - 2xf'(x) - f'(x) = 6 - 2x\left[f(x)\right]^3 + 2\left[f(x)\right]

Factor out \( f'(x) \)

\[ f'(x)\left[3x^2\left[f(x)\right]^2 - 2x - 1\right] = 6 - 2x\left[f(x)\right]^3 + 2\left[f(x)\right] \]

Divide by the factor

\[ f'(x) = \frac{6 - 2x\left[f(x)\right]^3 + 2\left[f(x)\right]}{3x^2\left[f(x)\right]^2 - 2x - 1} \]

Replace \( f(x) \) by \( y \) and \( f'(x) \) by \( y' \)

\[ y' = \frac{6 - 2xy^3 + 2y}{3x^2y^2 - 2x - 1} \]

Notice \( y' \) is in terms of \( x \) and \( y \)

Find the slope of the tangent line to the curve \( x^2y^3 - 2xy = 6x + y + 19 \) at the point \((-2,1)\).

\[ y' = \frac{6 - 2xy^3 + 2y}{3x^2y^2 - 2x - 1} \]

we need \( y'(-2,1) \).

\[ y'(-2,1) = \frac{6 - 2(-2)(1)^3 + 2(1)}{3(-2)^2(1)^2 - 2(-2) - 1} = \frac{6 + 4 + 2}{12 + 4 - 1} = \frac{12}{15} = \frac{4}{5} \]
Notation:

\[ \frac{\Delta y}{\Delta x} = \text{the slope of the secant line} \]

\[ \frac{dy}{dx} = \text{the slope of the tangent line} = \text{the derivative} \quad \text{(this is called ‘Leibniz notation’)} \]

\[ \frac{d}{dx} \] can be thought of as an operator that tells you to take the action of taking the derivative (with respect to \( x \)) of whatever is to follow.

\[ \frac{dy}{dx} = \frac{d}{dx} \left( y \right) = \text{the derivative of } y \text{ with respect to } x \]

\[ \frac{d}{dx} \left( \frac{1}{2} \sin x + 15x^2 - \cos (3x) \right) = \frac{1}{2} \cos x + 30x + 3 \sin (3x) \]

Alternative way of solving the problem:

Find the slope of the tangent line to the curve \( x^2y^3 - 2xy = 6x + y + 19 \) at the point \((-2,1)\).

\[ x^2y^3 - 2xy = 6x + y + 19 \]

\[ (2x)y^3 + x^2 \left( 3y^2 \cdot \frac{dy}{dx} \right) - 2y - 2x \frac{dy}{dx} = 6 + \frac{dy}{dx} \]

Now just plug in \( x = -2 \) and \( y = 1 \)

\[ 2(-2)(1)^3 + (-2)^2 \left( 3(1)^2 \cdot \frac{dy}{dx} \right) - 2(1) - 2(-2) \frac{dy}{dx} = 6 + \frac{dy}{dx} \]

\[ -4 + 12 \frac{dy}{dx} - 2 + 4 \frac{dy}{dx} = 6 + \frac{dy}{dx} \]

\[ -6 + 16 \frac{dy}{dx} = 6 + \frac{dy}{dx} \]

\[ 15 \frac{dy}{dx} = 12 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{12}{15} = \frac{4}{5} \]
Spring 2008 Final Exam Question
6. Find the equation of the tangent line to $9x^2 + 16y^2 = 52$ through (2, -1).
(a) $-9x + 8y - 26 = 0$
(b) $9x - 8y - 26 = 0$
(c) $9x - 8y - 106 = 0$
(d) $8x + 9y - 17 = 0$
(e) $9x + 16y - 2 = 0$

$$9x^2 + 16y^2 = 52$$

$$18x + 32y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-18x}{32y}$$

$$\left. \frac{dy}{dx} \right|_{(2, -1)} = \frac{-18(2)}{32(-1)} = \frac{36}{32} = \frac{9}{8}$$

$$m = \frac{9}{8} (2, -1) \quad y = mx + b$$
$$-1 = \frac{9}{8}(2) + b \quad \Rightarrow b = -\frac{9}{4}$$

$$8y = 9x - 26 \quad 9x - 8y - 26 = 0$$

Fall 2008 Final Exam Question
4. What is the equation of the tangent line to the curve $x^3 + 2y^2 + 3xy = 6$ at the point (1,1) ?

A) $y = -\frac{6}{5}x + \frac{11}{5}$
B) $y = -3x + 4$
C) $y = -2x + 3$

D) $y = -\frac{6}{7}x + \frac{13}{7}$
E) $y = -\frac{6}{5}x + \frac{13}{5}$
F) $y = -4x + 1$

$x^3 + 2y^2 + 3xy = 6$

$$3x^2 + 4y \frac{dy}{dx} + 3y + 3x \frac{dx}{dx} = 0$$

at the point (1,1)

$$3(1)^2 + 4(1) \frac{dy}{dx} + 3(1) + 3(1) \frac{dy}{dx} = 0$$

$$7 \frac{dy}{dx} = -6$$

$$\frac{dy}{dx} = \frac{-6}{7}$$

$m = \frac{-6}{7} (1,1)$

$$y = mx + b$$

$$1 = \frac{-6}{7}(1) + b \quad \Rightarrow b = \frac{13}{7}$$

$$y = \frac{-6}{7}x + \frac{13}{7}$$