4.8 Anti-Derivatives

Before: Given a function find its derivative
Now: Given a function’s derivative, find the function

\[ f'(x) = 8 \]

\[ f(x) = 8x + C \]

constant

**Reverse Power Rule**

\[ f'(x) = x^n \]

\[ f(x) = \frac{x^{n+1}}{n+1} + C \]

for all \( n \neq -1 \)

\[ f'(x) = x^4 \]

\[ f(x) = \frac{x^5}{5} + C \]

\[ f'(x) = 3x^2 + 2x^3 - 5 \]

\[ f(x) = \frac{3x^3}{3} + \frac{2x^4}{4} - 5x + C \]

\[ f'(x) = x^3 + \frac{x^4}{2} - 5x + C \]

\[ f'(x) = \sin x - 3\sec^2 x \quad \Rightarrow \quad f(x) = -\cos x - 3\tan x + C \]
\[ f'(x) = \sqrt{x} + x^2 - \frac{1}{x^2} = x^{1/2} + x^2 - x^{-2} \]

\[ f(x) = \frac{x^{3/2}}{3/2} + \frac{x^3}{3} - \frac{x^{-1}}{-1} + C = \frac{2x^{3/2}}{3} + \frac{x^3}{3} + \frac{1}{x} + C \]

If given a point on the function, we can solve for \( C \).

\[ f(4) = 30 \]

\[ f(4) = \frac{2 \cdot 4^{3/2}}{3} + \frac{4^3}{3} + \frac{1}{4} + C \overset{\text{set}}{=} 30 \Rightarrow \frac{16}{3} + \frac{64}{3} + \frac{1}{4} + C = 30 \]

\[ \Rightarrow C = 30 - \frac{80}{3} - \frac{1}{4} = \frac{360 - 320 - 3}{12} = \frac{37}{12} \]

\[ f(x) = \frac{2x^{3/2}}{3} + \frac{x^3}{3} + \frac{1}{x} + \frac{37}{12} \]
\[ f''(x) = 2x^3 + 3x^2 - 4x + 5 \quad f(0) = 2 \quad f(1) = 0 \]

\[ f'(x) = 2 \frac{x^4}{4} + 3 \frac{x^3}{3} - 4 \frac{x^2}{2} + 5x + C \]

\[ f'(x) = \frac{x^4}{2} + x^3 - 2x^2 + 5x + C \]

\[ f(x) = \frac{x^5}{10} + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + Cx + D \]

\[ f(0) = 2 \implies D = 2 \]

\[ f(1) = 0 \]

\[ f(1) = \frac{1}{10} + \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + C + 2 = 0 \implies \frac{6 + 15 - 40 + 150 + 120}{60} + C = 0 \implies C = \frac{-251}{60} \]

\[ f(x) = \frac{x^5}{10} + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - \frac{251}{60} x + 2 \]
A car braked with a constant deceleration of 16 ft./sec². producing skid marks measuring 200 ft. before coming to a stop. How fast was the car traveling when the brakes were first applied?

\[ a(t) = -16 \quad \text{The anti-derivative of acceleration is velocity.} \]

\[ v(t) = -16t + C \quad \text{We call the beginning of the skid } t = 0. \text{ We need to find the time it takes to get to the end of the skid.} \leftarrow v = 0 \]

\[ 0 = -16t + C \]

\[ \Rightarrow t = \frac{C}{16} \quad \text{The anti-derivative of velocity is position.} \]

\[ s(t) = -8t^2 + Ct + D \quad \text{At the beginning of the skid } s = 0. \leftarrow s(0) = 0 \]

\[ s(t) = -8t^2 + Ct \quad \text{At the end of the skid } s = 200. \leftarrow s\left(\frac{C}{16}\right) = 200 \]

\[ s\left(\frac{C}{16}\right) = -8\left(\frac{C}{16}\right)^2 + C\left(\frac{C}{16}\right) = 200 \]

\[ -8\left(\frac{C^2}{16\cdot16}\right) + \left(\frac{C^2}{16}\right) = 200 \]

\[ -\frac{C^2}{32} + \frac{C^2}{16} = 200 \]

\[ \frac{C^2}{32} = 200 \quad \Rightarrow \quad C^2 = 6400 \quad C = 80 \]

\[ v(t) = -16t + 80 \]

\[ v(0) = 80 \text{ ft./sec.} \]

\[ v(0) \approx 54.54 \text{ mph.} \]
<table>
<thead>
<tr>
<th>Function</th>
<th>General antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{1}{n+1} x^{n+1} + C, ; n \neq -1$</td>
</tr>
<tr>
<td>$\sin kx$</td>
<td>$-\frac{1}{k} \cos kx + C$</td>
</tr>
<tr>
<td>$\cos kx$</td>
<td>$\frac{1}{k} \sin kx + C$</td>
</tr>
<tr>
<td>$\sec^2 kx$</td>
<td>$\frac{1}{k} \tan kx + C$</td>
</tr>
<tr>
<td>$\csc^2 kx$</td>
<td>$-\frac{1}{k} \cot kx + C$</td>
</tr>
<tr>
<td>$\sec kx \tan kx$</td>
<td>$\frac{1}{k} \sec kx + C$</td>
</tr>
<tr>
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<td>$-\frac{1}{k} \csc kx + C$</td>
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<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1 - k^2x^2}}$</td>
<td>$\frac{1}{k} \sin^{-1} kx + C$</td>
</tr>
<tr>
<td>$\frac{1}{1 + k^2x^2}$</td>
<td>$\frac{1}{k} \tan^{-1} kx + C$</td>
</tr>
<tr>
<td>$\frac{1}{x \sqrt{k^2x^2 - 1}}$</td>
<td>$\sec^{-1} kx + C, ; kx &gt; 1$</td>
</tr>
<tr>
<td>$a^{kx}$</td>
<td>$\left(\frac{1}{k \ln a}\right) a^{kx} + C, ; a &gt; 0, ; a \neq 1$</td>
</tr>
</tbody>
</table>
\[ f'(x) = e^{3x} + e^{\frac{x}{2}} + x^e + x^{\sqrt{2}-1} + 4^x \]

\[ f(x) = \frac{1}{3} e^{3x} + \frac{1}{2} e^{\frac{x}{2}} + \frac{x^{e+1}}{e+1} + \frac{x^{\sqrt{2}-1+1}}{\sqrt{2}-1+1} + \frac{1}{\ln 4} 4^x + C \]

\[ f(x) = \frac{1}{3} e^{3x} + 2e^{\frac{x}{2}} + \frac{x^{e+1}}{e+1} + \frac{x^{\sqrt{2}}}{\sqrt{2}} + \frac{1}{\ln 4} 4^x + C \]
A special symbol is used to denote the collection of all antiderivatives of a function $f$. 

**DEFINITION**  The collection of all antiderivatives of $f$ is called the **indefinite integral** of $f$ with respect to $x$, and is denoted by 

$$\int f(x) \, dx.$$ 

What function has $f(x)$ as its deriv.?

The symbol $\int$ is an **integral sign**. The function $f$ is the **integrand** of the integral, and $x$ is the **variable of integration**.

After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. We will have more to say about why this is important in Chapter 5.

**FIGURE 4.51** The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$. 

![Graph showing curves of $y = x^3 + C$]
\[
\int x^{-3} (x + 1) \, dx
= \int (x^{-2} + x^{-3}) \, dx
\]
\[
= \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C
= \frac{-1}{x} + \frac{-1}{2x^2} + C
\]
\[
\int \frac{4 + \sqrt{t}}{t^3} \, dt
\]
\[
= \int \left( \frac{4}{t^3} + \frac{\sqrt{t}}{t^3} \right) \, dt
\]
\[
= \int \left( 4t^{-3} + t^{\frac{1}{2}-3} \right) \, dt
= \int \left( 4t^{-3} + t^{\frac{-5}{2}} \right) \, dt
\]
\[
= 4 \cdot \frac{t^{-2}}{-2} + \frac{t^{-\frac{3}{2}}}{-\frac{3}{2}} + C
= \frac{-2}{t^2} - \frac{2}{3t^{\frac{3}{2}}} + C
\]
Solution (Integral) Curves

Exercises 115–118 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

118. \[ \frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \pi \sin \pi x \]

\[ y = \frac{1}{2} \sqrt{x} + \pi \sin (\pi x) + c \]

\[ \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} + \pi \sin (\pi x) \]

\[ \text{deriv.} \quad y = \frac{1}{2} \cdot \frac{x^{\frac{1}{2}} + 1}{\frac{1}{2} + 1} + \pi \cdot \frac{1}{\pi} - \cos (\pi x) + C \]

\[ y = \frac{1}{2} x^{\frac{1}{2}} - \cos (\pi x) + C \]

\[ y = \sqrt{x} - \cos (\pi x) + C \]

\[ y = \sqrt{x} - \cos (\pi x) + C \]