5.1 The Area Problem

**Goal:** To find the area under the graph of $f(x)$ and above the $x$-axis between $x = a$ and $x = b$

**Problem:** curved sides

**How:** Use rectangles to approximate the area

$f(x) = x^2$ on the interval $[1, 5]$

using 4 rectangles and the right endpoint of each interval to get the height of each rectangle

Area $= f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 + f(5) \cdot 1$

Area $\approx 4 + 9 + 16 + 25$

Area $\approx 54$ units$^2$
\( f(x) = x^2 \) on the interval \([1, 5]\)
using 4 rectangles
and the left endpoint
of each interval to get
the height of each rectangle
Area = \( f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 \)
Area \( \approx 1 + 4 + 9 + 16 \)
Area \( \approx 30 \text{ units}^2 \)

\( f(x) = x^2 \) on the interval \([1, 5]\)
using 4 rectangles
and the midpoint
of each interval to get
the height of each rectangle
Area = \( f(\frac{3}{2}) \cdot 1 + f(\frac{5}{2}) \cdot 1 + f(\frac{7}{2}) \cdot 1 + f(\frac{9}{2}) \cdot 1 \)
Area \( \approx \frac{9}{4} + \frac{25}{4} + \frac{49}{4} + \frac{81}{4} \)
Area \( \approx \frac{164}{4} \) units\(^2\)
Area \( \approx 41 \text{ units}^2 \)
\( f(x) = x^2 \) on the interval \([1, 5]\) using 8 rectangles and the right endpoint of each interval to get the height of each rectangle.

\[
\begin{align*}
\text{Area} & \approx \left( \frac{9}{4} + \frac{4}{4} + \frac{25}{4} + 9 + \frac{49}{4} + 16 + \frac{81}{4} + 25 \right) \\
\text{Area} & \approx \frac{1}{2} \left( \frac{164}{4} + 54 \right) \text{ units}^2 \\
\text{Area} & \approx \frac{95}{2} \text{ units}^2 \\
\text{Area} & \approx 47.5 \text{ units}^2
\end{align*}
\]
5.1 The Area Problem
5.3 The Definite Integral

An Approximation of the Integral of
\( f(x) = x^2 \)
on the Interval [1, 5]
Using a Right-endpoint Riemann Sum

Areas: 45 24000000

Partitions: 10

An Approximation of the Integral of
\( f(x) = x^2 \)
on the Interval [1, 5]
Using a Right-endpoint Riemann Sum

Areas: 43 76000000

Partitions: 20
An Approximation of the Integral of $f(x) = x^2$
on the interval $[1, 5]$
Using a Right-endpoint Riemann Sum

Area: 42.540800000

Points: 48

An Approximation of the Integral of $f(x) = x^2$
on the interval $[1, 5]$
Using a Right-endpoint Riemann Sum

Area: 41.935000000

Points: 80
5.1 The Area Problem

5.3 The Definite Integral

as the number of rectangles increases, accuracy increases.

An Approximation of the Integral of
\( f(x) = x^2 \)
on the interval \([1, 5]\)
Using a Right-endpoint Riemann Sum

Area: 41.6374994

Use \( n = 4 \) rectangles and left endpoints to estimate

\[ \int_1^5 \frac{2-x}{x} \, dx \]

The area under the graph of \( f(x) \) and above the \( x \)-axis from \( x=1 \) to \( x=5 \)

\[ f(1) = 2 - 1 = 1 \]
\[ f(2) = \frac{2}{2} - 2 = 0 \]
\[ f(3) = \frac{2}{3} - 3 = -1 \frac{1}{3} \]
\[ f(4) = \frac{2}{4} - 4 = -1 \frac{1}{2} \]
\[ f(5) = \frac{2}{5} - 5 = -1 \frac{1}{5} \]

Approximate:

\[ \sum_{i=1}^{n} f(x_i) \Delta x \]

\[ \approx \left( \frac{1}{5} + \frac{4}{30} + \frac{9}{30} + \frac{14}{30} \right) \]

\[ \approx \left( \frac{1}{5} + \frac{1}{2} + \frac{1}{2} \right) \]

\[ \approx \left( \frac{6 - 2}{5} \right) \]
5.3 The Definite Integral

**Definition of a Definite Integral** If \( f \) is a function defined for \( a \leq x \leq b \), we divide the interval \([a, b]\) into \( n \) subintervals of equal width \( \Delta x = (b - a)/n \). We let \( x_0 (= a), x_1, x_2, \ldots, x_n (= b) \) be the endpoints of these subintervals and we let \( x_1^*, x_2^*, \ldots, x_n^* \) be any sample points in these subintervals, so \( x_i^* \) lies in the \( i \)th subinterval \([x_{i-1}, x_i]\). Then the definite integral of \( f \) from \( a \) to \( b \) is

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

provided that this limit exists. If it does exist, we say that \( f \) is integrable on \([a, b]\).

\[
\int_a^b f(x) \, dx
\]

**Definite Integral**

**Note:** The symbol \( \int \) was introduced by Leibniz and is called an integral sign. It is an elongated \( S \) and was chosen because an integral is a limit of sums. In the notation \( \int_a^b f(x) \, dx \), \( f(x) \) is called the integrand and \( a \) and \( b \) are called the limits of integration; \( a \) is the lower limit and \( b \) is the upper limit. For now, the symbol \( dx \) has no meaning by itself; \( \int_a^b f(x) \, dx \) is all one symbol. The \( dx \) simply indicates that the independent variable is \( x \). The procedure of calculating an integral is called integration.
5.2 Riemann Sum

\[ \lim_{n \to \infty} \sum_{i=1}^{n} f \left( x_i^* \right) \Delta x \]

\begin{align*}
& a & a + \Delta x & a + 2\Delta x & a + 3\Delta x & a + i\Delta x \\
& x_0 & x_1 & x_2 & x_3 & x_i
\end{align*}

using right endpoints we can simplify the Riemann sum

\[ \lim_{n \to \infty} \Delta x \left[ f \left( x_1 \right) + f \left( x_2 \right) + \cdots + f \left( x_i \right) + \cdots + f \left( x_n \right) \right] \]

\[ x_i^* = x_i \quad \text{and} \quad x_i = a + i\Delta x \]

\[ \Delta x = \frac{b-a}{n} \]

\[ \lim_{n \to \infty} \sum_{i=1}^{n} f \left( x_i^* \right) \Delta x = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f \left( a + \frac{b-a}{n} \cdot i \right) \]

Appendix E : Sigma Notation page A37

3 THEOREM  Let \( c \) be a constant and \( n \) a positive integer. Then

(a) \( \sum_{i=1}^{n} 1 = n \)  \hspace{1cm} (b) \( \sum_{i=1}^{n} c = nc \)

(c) \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)  \hspace{1cm} (d) \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \)

(e) \( \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2 \)
\[ f(x) = x^2 \text{ on the interval } [1, 5] \]

Find the exact area using the Riemann sum

\[
\int_1^5 x^2 \, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f \left( a + \frac{b-a}{n} \cdot i \right)
\]

\[
\int_1^5 x^2 \, dx = \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \left( 1 + \frac{4}{n} \cdot i \right) = \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \left( 1 + \frac{4}{n} \cdot i + \frac{16}{n^2} \cdot i^2 \right)
\]

\[
= \lim_{n \to \infty} \frac{4}{n} \left[ \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \frac{4}{n} \cdot i + \sum_{i=1}^{n} \frac{16}{n^2} \cdot i^2 \right] = \lim_{n \to \infty} \frac{4}{n} \left[ \sum_{i=1}^{n} 1 + \frac{8}{n^2} \sum_{i=1}^{n} i + \frac{16}{n^2} \sum_{i=1}^{n} i^2 \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{4 + 8 \frac{n(n+1)}{2} + 16 \frac{n(n+1)(2n+1)}{6}}{n^2} \right] = \lim_{n \to \infty} \left[ \frac{4 + 32 \frac{n(n+1)}{2} + 64 \frac{n(n+1)(2n+1)}{6}}{n^2} \right]
\]

\[
= \lim_{n \to \infty} \left[ 4 + 16 \frac{n^2}{n^2} + \frac{64}{n} + \frac{32}{n^2} \cdot \frac{1}{n^2} \cdot \frac{2n^3 + 3n + 1}{3} \right] = \lim_{n \to \infty} \left[ 4 + 16 + \frac{60}{3} + \frac{64}{3} \cdot \frac{1}{3} \right] = \frac{124}{3}
\]

Too much work. We find an easier way in section 5.3

\[ = \frac{41}{3} \]
Not all functions are integrable

**Theorem** If \( f \) is continuous on \([a, b]\), or if \( f \) has only a finite number of jump discontinuities, then \( f \) is integrable on \([a, b]\); that is, the definite integral \( \int_a^b f(x) \, dx \) exists.

Area under the x-axis is considered negative
the definite integral measures net area

**Figure 3** \( \sum f(x_i) \Delta x \) is an approximation to the net area

**Figure 4** \( \int_a^b f(x) \, dx \) is the net area
5.1 The Area Problem

5.3 The Definite Integral

**Properties**:

1. \( \int_{a}^{a} f(x) \, dx = 0 \)

2. \( \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \)

3. \( \int_{a}^{b} [f(x) \pm g(x)] \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \)

4. \( \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \)

5. \( \int_{a}^{b} c \, dx = c(b-a) \)

6. If \( a < c < b \), then \( \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \)

**Examples**:

i. \( \int_{0}^{2} f(x) \, dx = 4 \)

ii. \( \int_{0}^{5} f(x) \, dx = \int_{0}^{2} f(x) \, dx + \int_{2}^{5} f(x) \, dx = 4 + 6 = 10 \)

iii. \( \int_{0}^{5} f(x) \, dx = -3 \)

iv. \( \int_{0}^{9} f(x) \, dx = \int_{0}^{5} f(x) \, dx + \int_{5}^{9} f(x) \, dx = 10 + (-8) = 2 \)
Evaluate the integral by interpreting it in terms of areas.

\[ \int_{0}^{5} \left( 1 + \sqrt{25 - x^2} \right) dx = \frac{1}{4} \pi r^2 + 5 \]

\[ = \frac{25\pi}{4} + 5 \]

- The function \( y = 1 + \sqrt{25 - x^2} \) represents a quarter circle centered at (0,1) with radius 5.
- Only the right upper quarter circle is considered.

\[ \int_{0}^{12} |x - 6| dx = 36 \]

- The function \( y = |x - 6| \) is a triangle shifted 6 units to the right.
- The area under the curve is calculated as 36.