5.4 The Fundamental Theorem of Calculus

Goal:
To find the area under the graph of \( f(x) \)
above the \( x \)-axis between \( x = a \) and \( x = b \).
This is represented by the symbol:
\[
\int_{a}^{b} f(x) \, dx
\]
This is called a definite integral.

The Fundamental Theorem of Calculus, Part 2

If \( f \) is continuous on \([a, b]\), then
\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]
Where \( F \) is the antiderivative of \( f \).

\( f(x) = x^2 \). Find its antiderivative \( F(x) = \frac{x^3}{3} + C \)

\[
\int_{1}^{5} x^2 \, dx = F(5) - F(1) = \left( \frac{5^3}{3} + C \right) - \left( \frac{1^3}{3} + C \right)
\]
\[
= \frac{125}{3} - \frac{1}{3} + C - C = \frac{124}{3}
\]
The \( C \) will cancel each time
so we will leave it out from here on.
\[ \int_{-2}^{2} (x^3 - 2x + 3) \, dx \quad f(x) = x^3 - 2x + 3. \text{ Find its antiderivative.} \]

\[ F(x) = \frac{x^4}{4} - x - 3x \]

\[ F(2) = \frac{2^4}{4} - 2^2 + 3(2) = 4 - 4 + 6 = 6 \]

\[ F(-2) = \frac{(-2)^4}{4} - (-2)^2 + 3(-2) = 4 - 4 - 6 = -6 \]

\[ \int_{0}^{\pi} \sin x \, dx \quad f(x) = \sin x. \text{ Find its antiderivative.} \]

\[ F(x) = -\cos x \]

\[ F(\pi) = -\cos \pi \]

\[ F(0) = -\cos 0 \]

\[ F(\pi) = 1 \]

\[ F(0) = 1 \]

\[ \int_{0}^{\pi} \sin x \, dx = F(\pi) - F(0) = 1 - (-1) = 2 \]
\[
\int_{\frac{1}{\sqrt{2}}}^{1} \left(\frac{x^7}{2} - \frac{1}{x^5}\right) \, dx \quad f(x) = \frac{x^7}{2} - \frac{1}{x^5}.
\]
Find its antiderivative.

\[
= -\int_{\frac{1}{\sqrt{2}}}^{1} \left(\frac{x^7}{2} - \frac{1}{x^5}\right) \, dx
\]

since $\sqrt{2} > 1$

\[
-\int_{\frac{1}{\sqrt{2}}}^{1} \left(\frac{x^7}{2} - \frac{1}{x^5}\right) \, dx = -\left[ F\left(\sqrt{2}\right) - F\left(1\right) \right]
\]

\[
= -\left[ \frac{17 - 5}{16} \right] = -\frac{12}{16} = -\frac{3}{4}
\]

\[
F\left(\sqrt{2}\right) = 16 + \frac{1}{4(\sqrt{2})^4}
\]

\[
= 16 + \frac{1}{4 \cdot 4} = \frac{17}{16}
\]

\[
F\left(1\right) = \frac{1}{16} + \frac{1}{4 \cdot 4}
\]

\[
= \frac{1}{16} + \frac{1}{4} = \frac{5}{16}
\]
5.4 The Fundamental Theorem of Calculus

\[ \int_0^x e^t \, dt \]

\[ F(x) = e^x \]

\[ F(1) = e^1 = e \]

\[ F(0) = e^0 = 1 \]

\[ F(1) - F(0) = e - 1 \]

Answer: \( e - 1 \)

\[ \int_{\ln 2}^x 7 \, dt \]

\[ F(x) = 7x \]

\[ F(\ln 7) = 7 \ln 7 = 7 \]

\[ F(\ln 4) = 7 \ln 4 = 4 \]

\[ F(\ln 7) - F(\ln 4) = 7 - 4 \]

Answer: 3

\[ \int \frac{1}{x} \, dx \]

\[ F(x) = \ln x \]

\[ F(4) = \ln 4 \]

\[ F(3) = \ln 3 \]

\[ F(4) - F(3) = \ln 4 - \ln 3 \]

\[ \ln \frac{4}{3} \]

\[ \int_{\frac{1}{e}}^e \frac{1}{x} \, dx \]

\[ F(x) = \ln x \]

\[ F(e^2) = \ln e^2 = 4 \]

\[ F(e) = \ln e = 1 \]

\[ F(e^2) - F(e) = 4 - 1 \]

Answer: \( \frac{3}{2} \)
The Fundamental Theorem of Calculus, Part 1

If \( f \) is continuous on \([a,b]\), then the function \( g \) defined by

\[
    g(x) = \int_{a}^{x} f(t) \, dt
\]

is continuous on \([a,b]\), differentiable on \((a,b)\) and \( g'(x) = f(x) \)

\[
    g(x) = \int_{1}^{x} \sqrt{3t-t^2} \, dt
    \]

\[
    g'(x) = \frac{d}{dx} \left( \int_{1}^{x} \sqrt{3t-t^2} \, dt \right) = \sqrt{3x-x^2}
\]
\[ g(x) = \int_{2}^{3x^3} \frac{4}{\sqrt{3t^2 + x^3}} \, dt \quad \Rightarrow \quad g'(x) = \frac{4}{\sqrt{3x^2 + (3x^2)^3}} \cdot (6x) \]

**Why?**

\[
g'(x) = \frac{d}{dx} \left( \int_{2}^{3x^3} \frac{4}{\sqrt{3t^2 + x^3}} \, dt \right) = \frac{d}{dx} \left( \int_{u}^{3x^3} \frac{4}{\sqrt{3t^2 + x^3}} \, dt \right) = \frac{d}{du} \left( \int_{u}^{3x^3} \frac{4}{\sqrt{3t^2 + x^3}} \, dt \right) \frac{du}{dx}
\]

Let \( u = 3x^2 \)

\[
\frac{du}{dx} = 6x
\]

\[
g'(x) = \frac{24x}{x\sqrt{3 + 27x^8}} = \frac{24}{\sqrt{3 + 27x^8}}
\]

simplified version

\[
g'(x) = -\frac{\sqrt{x}}{(\sqrt{x})^2 + 1} \cdot \left( \frac{1}{2\sqrt{x}} \right) + \frac{\tan x}{(\tan x)^2 + 1} = -\frac{1}{2(x+1)} + \tan x
\]
13. The graph of \( f \) below consists of line segments and semicircles. Let \( g(x) = \int_0^x f(t) \, dt \).

Answer the following questions.

(a) \( g(14) = 12 + \frac{9\pi}{2} + 2 - 2 - 8\pi = \frac{4 + \frac{5\pi}{2}}{} \)

(b) \( g(10) = 12 + \frac{9\pi}{2} + 2 - 2 = \frac{12 + \frac{9\pi}{2}}{} \)

(c) \( g'(6) = f(6) = \frac{2}{2} \)

(d) What is the absolute maximum value of \( g \) on the interval [0,14]?