5.5/5.6 Substitution

THE SUBSTITUTION RULE If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x)) g'(x) \, dx = \int f(u) \, du
\]

inside function \( = \) derivative of the inside function (or some multiple of it)

outside function

Integration by Substitution: Evaluating \( \int f(g(x)) g'(x) \, dx \)

Step 1 Let \( u = g(x) \), where \( g(x) \) is part of the integrand, usually, the “inside function” of the composite function \( f(g(x)) \).

Step 2 Compute \( du = g'(x) \, dx \).

Step 3 Use the substitution \( u = g(x) \) and \( du = g'(x) \, dx \) to transform the integral into one that involves only \( u \) : \( \int f(u) \, du \).

Step 4 Find the resulting integral.

Example 1

\[
\int_0^2 x^2 \sqrt{1 + x^3} \, dx
\]

\[
= \int_1^9 u \cdot \frac{1}{3} \, du = \frac{1}{3} \int_1^9 u^{1/2} \, du
\]

\[
= \frac{1}{3} \cdot \frac{2}{3} \left[u^{3/2}\right]_1^9 = \frac{2}{9} \left(9^{3/2} - 1^{3/2}\right)
\]

\[
= \frac{2}{9} \left(9\sqrt{9} - 1\right) = \frac{2}{9} \left(27 - 1\right) = \frac{52}{9}
\]

Option 1: Limit Switch

\[
x = 2 \Rightarrow u = 9
\]

\[
x = 0 \Rightarrow u = 1
\]
example 2

\[
\int_{\pi/6}^{\pi/4} \csc^2 x \cot x \, dx = \int u(-du)
\]

\[-\int u \, du = -\frac{u^2}{2}\]

\[
= \left[ -\frac{1}{2} \cot^2 x \right]_{\pi/6}^{\pi/4} = -\frac{1}{2} \left[ \left( \frac{\cos x}{\sin x} \right)^2 \right]_{\pi/6}^{\pi/4}
\]

\[
= -\frac{1}{2} \left[ \left( \frac{\cos (\pi/4)}{\sin (\pi/4)} \right)^2 - \left( \frac{\cos (\pi/6)}{\sin (\pi/6)} \right)^2 \right] = -\frac{1}{2} [1 - 3] = 1
\]

now substitute back into \( u = \cot x \)

Option 2: Don't Limit Switch

\[
tan x = \frac{\sin x}{\cos x} \quad \text{so} \quad cot x = \frac{\cos x}{\sin x}
\]

\[
\frac{\cos (\pi/6)}{\sin (\pi/6)} = \frac{\sqrt{3}}{2} \quad \frac{\cos (\pi/4)}{\sin (\pi/4)} = \frac{\sqrt{2}}{2}
\]

\[
\frac{\cos (\pi/6)}{\sin (\pi/6)} = \frac{\sqrt{3}}{2} \quad \frac{\cos (\pi/4)}{\sin (\pi/4)} = \frac{\sqrt{2}}{2} = \sqrt{3}
\]

example 3

without substitution

\[
\int_0^1 \sin(3\pi x) \, dx = \frac{-1}{3} \cos(3\pi x) + C
\]

\[
\cos(3\pi) = \cos \pi
\]

\[
\int_0^1 \cos(3\pi x) \, dx = F(1) - F(0)
\]

\[
\int_0^1 \cos(3\pi x) \, dx = \frac{-1}{3\pi} \left[ \cos(3\pi) - \cos(0) \right]
\]

\[
= \frac{-1}{3\pi} \left[ -2 \right] = \frac{2}{3\pi}
\]
**Example 3**

With substitution \( u = 3\pi x \)

\[
\int_{0}^{1} \sin(3\pi x) \, dx = \int \sin u \cdot \frac{1}{3\pi} \, du = \frac{-\cos 0 - (-\cos 0)}{3\pi} = \frac{2}{3\pi}
\]

**Example 4**

\[
\int \frac{1}{\sqrt{x} \left(\sqrt{x} + 1\right)^2} \, dx = \int \frac{1}{u^2} \, du = \int u^{-2} \, du = -\frac{2}{\sqrt{x} + 1} + C
\]
\[
\ln 2 \int \frac{6e^{3x}}{dx} = \int 6e^{3x} dx \\
\text{antideriv} = 6 \left[ \frac{e^{3x}}{3} \right] = 2 \left[ e^{3x} \right] \\
\ln 2 \left( \frac{e^{3(0)}}{3} \right) = 2 \left( e^0 \right) \\
= 2 (1) = 2
\]
example 5

\[ \int_5^x \frac{dx}{\sqrt{x-1}} \quad u = x - 1 \quad \Rightarrow \quad x = u + 1 \quad \text{if} \quad x = 5 \Rightarrow u = 4 \]
\[ \int_2^x \frac{dx}{\sqrt{x-1}} \quad d\,u = dx \quad \text{if} \quad x = 2 \Rightarrow u = 1 \]

\[ \int_2^5 \frac{x}{\sqrt{x-1}} \, dx = \int_1^4 \left( \frac{u+1}{\sqrt{u}} \right) \, du = \int_1^4 \left( \frac{u^{3/2}}{3} + \frac{u^{1/2}}{1} \right) \, du = \left[ \frac{2}{3} u^{3/2} + 2\sqrt{u} \right]^4_1 \]
\[ = \left( \frac{2}{3} \cdot 4^{3/2} + 2\sqrt{4} \right) - \left( \frac{2}{3} \cdot 1^{3/2} + 2\sqrt{1} \right) = \left( \frac{2}{3} \cdot 8 + 2 \cdot 2 \right) - \left( \frac{2}{3} + 2 \right) \]
\[ = \frac{16}{3} - \frac{2}{3} + 4 - 2 = \frac{14}{3} + 2 = \frac{20}{3} \]

Integrals of Odd and Even Functions

Suppose that \( f \) is continuous on \([-a, a] \).

a. If \( f \) is even, then \( \int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx \).

\[ \int_{-\pi}^{\pi} (1 + x^2 + \cos x) \, dx = 2 \int_0^{\pi} (1 + x^2 + \cos x) \, dx = 2 \left( x + \frac{x^3}{3} - \sin x \right)_0^\pi = 2 \left( \pi + \frac{\pi^3}{3} \right) \]

b. If \( f \) is odd, then \( \int_{-a}^a f(x) \, dx = 0 \).

\[ \int_{-2}^2 (x^3 - \sin x) \, dx = 0 \]