### Section 2.2 Introduction To Limits

\[ \lim_{x \to a} f(x) = L \]

"the limit as \( x \) approaches \( a \) (from either side)
of the function \( f(x) \) is \( L \)"

means as \_________________, the
function values \______________

It actually doesn't matter

\[ \lim_{x \to a^-} f(x) \]

\[ \lim_{x \to a^+} f(x) \]

\[ f(x) \text{ approaches} \]

\[ y = x^2 - x + 2 \]

**Example:**

\[ \lim_{x \to 1} \frac{x - 1}{x^2 - 1} \]

**One-sided Limits**

1. left-hand limit

\[ \lim_{x \to a^-} f(x) = L \]

2. right-hand limit

\[ \lim_{x \to a^+} f(x) = L \]

For a regular (2-sided) limit to exist, you must have:

\[ \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \]

Then you can say:

\[ \lim_{x \to a} f(x) = L \]

\[ \lim_{x \to a^-} g(x) = \lim_{x \to a^+} g(x) = \]

\[ \lim_{x \to a} g(x) = \lim_{x \to a^+} g(x) = \]

\[ \lim_{x \to a^-} g(x) = \lim_{x \to a^+} g(x) = \]

\[ \lim_{x \to a^+} g(x) = \lim_{x \to a^+} g(x) = \]

\[ \lim_{x \to a^-} g(x) = \lim_{x \to a^-} g(x) = \]

\[ \lim_{x \to a} g(x) = \lim_{x \to a} g(x) = \]
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The value of the limit can become \( \lim_{x \to a} f(x) = \) _______.

\[ f(x) = \]

the values of \( f(x) \) become _______ and _______ as \( x \) becomes closer and closer to \( a \).

the values of \( f(x) \) can be made arbitrarily large (as large as we please) by taking \( x \) _______.

The line \( x = a \) is then called a _______.

\[ \lim_{x \to a} - \frac{\cos x}{x^2} = \]

Use a table of values to estimate the value of the limit.

\[ \lim_{x \to 0} \frac{1 + x - e^x}{x^3} = \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
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</thead>
<tbody>
<tr>
<td>-1</td>
<td></td>
<td>-0.5</td>
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<td>-0.5</td>
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<td>0.5</td>
<td></td>
<td>1</td>
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</tbody>
</table>
What do you do when the denominator is 0?

e.x.1) \( \lim \frac{2-x}{x^2 + 3} \)

e.x.2) \( \lim \frac{x^3 + 13x^2 - 48x}{x^3 - 27} \)

Limit Laws

Let \( k \) and \( a \) be constants. Let \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) (with \( L \) and \( M \) finite and \( M \neq 0 \))

1) \( \lim_{x \to a} [f(x) + g(x)] = L + M \)

2) \( \lim_{x \to a} [f(x) - g(x)] = L - M \)

3) \( \lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M \)

4) \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \) (with \( M \neq 0 \))

5) \( \lim_{x \to a} kf(x) = k \cdot L \)

6) \( \lim_{x \to a} c = c \)

7) \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \)

8) \( \lim_{x \to a} [f(x)]^n = [L]^n \)

The combination of the limit laws allows one to find the value of a limit by plugging in the value of \( a \):

\[
\lim_{x \to a} \frac{(3-x)\sqrt{1-x}}{x^2 + x - 2} = \frac{\lim_{x \to a} (3-x) \cdot \lim_{x \to a} \sqrt{1-x}}{\lim_{x \to a} (x^2 + x - 2)}
\]

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\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

ex.1) \[ \lim_{x \to 3} \frac{\sqrt{x^2 + 3} - 2}{x - 1} \]

ex.2) \[ \lim_{x \to 2} \frac{x - 2}{x - 7} \]

ex.3) \[ \lim_{x \to 0} \frac{(x + h)^3 - 64}{h} \]

ex.4) \[ \lim_{x \to 3} \frac{x^2 - 3}{x - 3} \]

No algebra technique to cancel the division by zero.

Resort back to the estimation techniques used earlier.

\[ \lim_{x \to 3} \frac{x^2 - 3}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x - 3} \]

plug in \( x = 2.999 \)

\[ \lim_{x \to 3} \frac{x - 3}{x - 3} = \lim_{x \to 3} \frac{x - 3}{0} \]

plug in \( x = 3.001 \)

\[ \frac{2.999(2.999 - 3)}{2.999 - 3} = \frac{3.001(3.001 - 3)}{3.001 - 3} \]

\[ -2.999 \]

\[ -3.001 \]
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The Squeeze Theorem can be used to find the value of a very important limit.

If we let \( a \) be a constant, then a more general version of the limit above is

Finally a third fundamental limit based on the two limits above is

The three limits above can form the foundation for the proof of the fact that the derivative of \( \sin x \) is ________.

"Why should I care about the Squeeze Theorem?"

The Squeeze Theorem can be used to find the value of a very important limit.

If we let \( a \) be a constant, then a more general version of the limit above is

Finally a third fundamental limit based on the two limits above is

The three limits above can form the foundation for the proof of the fact that the derivative of \( \sin x \) is ________.