4.5 L’Hôpital’s Rule

\[
\lim_{x \to a} \frac{f(x)}{g(x)} \text{ when } f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a
\]

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{← this is called an indeterminate form}
\]

\[
\lim_{x \to a} \frac{f(x)}{g(x)} \text{ when } f(x) \to \pm\infty \text{ and } g(x) \to \pm\infty \text{ as } x \to a
\]

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \pm \frac{\infty}{\infty} \quad \text{← this is called an indeterminate form}
\]

These two types of indeterminate forms can be simplified using

**L’Hôpital’s Rule**

**L’HOSPITAL’S RULE** Suppose \( f \) and \( g \) are differentiable and \( g'(x) \neq 0 \) on an open interval \( I \) that contains \( a \) (except possibly at \( a \)). Suppose that

\[
\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0
\]

or that

\[
\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty
\]

(In other words, we have an indeterminate form of type \( \frac{0}{0} \) or \( \infty/\infty \).) Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

if the limit on the right side exists (or is \( \infty \) or \( -\infty \)).
\[
\lim_{x \to 1} \frac{x^3 - 1}{x^9 - 1}
\]

\[
\lim_{x \to 0} \frac{\sin(4x)}{\tan(5x)}
\]

\[
\lim_{x \to 0} \frac{x}{\arctan(8x)}
\]
\[ \lim_{x \to 0} \frac{e^{3x} - e^{-x} - 4x}{x - \sin x} \]

\[ \lim_{x \to \infty} \frac{(\ln x)^2}{x} \]
Other indeterminate forms

Need to be converted into \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \)

1. **Indeterminate products** "0 \( \pm \) \& + \infty"

Write the product as a quotient

\[
f \cdot g = \frac{f}{1} \quad \text{or} \quad f \cdot g = \frac{g}{1}
\]

\[
limit \ x \ tan \left( \frac{4}{x} \right)
\]

2. **Indeterminate differences** "\( \infty - \infty \)"

Convert the difference into a quotient

How?

a) factor out a common factor

b) find a common denominator

c) rationalize

\[
limit \ x e^{6x} - x
\]
\[
\lim_{x \to 0^+} \left( \frac{x}{x-1} - \frac{1}{\ln x} \right)
\]

Other indeterminate forms

Need to be converted into \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \)

3. **Indeterminate powers** \( \frac{0^0}{\infty^0} \) \( \frac{1^0}{\infty^0} \)

Use \( \ln \) to convert into a quotient

\[
\lim_{x \to a} f(x)^{g(x)}
\]

\[
y = \lim_{x \to a} f(x)^{g(x)} \implies \ln y = \ln \left[ \lim_{x \to a} f(x)^{g(x)} \right] \implies \ln y = \lim_{x \to a} \ln f(x)^{g(x)}
\]

let \( y \) equal the limit take the \( \ln \) of both sides interchange \( \ln \) and \( \lim \)

\[
\ln y = \lim_{x \to a} [g(x) \cdot \ln f(x)] \quad \leftarrow \text{this will be in the form } "0 \cdot \pm\infty"
\]

use the power rule for \( \ln \) follow the directions for \( 0 \cdot \infty \) to find the limit \( L \)

\[
\ln y = L \implies e^{\ln y} = e^L \implies y = e^L
\]

make both sides the exponent on \( e \) to get back to \( y \)
\[
\lim_{x \to 0} (1 - 6x)^{\frac{1}{2}}
\]

\[
\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx} \quad a, b \text{ real numbers}
\]