### 5.5/5.6 Substitution

**THE SUBSTITUTION RULE** If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x)) g'(x) \, dx = \int f(u) \, du
\]

Integration by Substitution: Evaluating \( \int f(g(x)) g'(x) \, dx \)

**Step 1** Let \( u = g(x) \), where \( g(x) \) is part of the integrand, usually, the “inside function” of the composite function \( f(g(x)) \).

**Step 2** Compute \( du = g'(x) \, dx \).

**Step 3** Use the substitution \( u = g(x) \) and \( du = g'(x) \, dx \) to transform the integral into one that involves only \( u \): \( \int f(u) \, du \).

**Step 4** Find the resulting integral.

---

**Example 1**

\[
\int_{0}^{2} x^2 \sqrt{1 + x^3} \, dx
\]

**Option 1: Limit Switch**

\[
x = 2 \implies u = \]

\[
x = 0 \implies u =
\]
example 2

\[
\int_{\pi/6}^{\pi/4} \csc^2 x \cot x \, dx
\]

\[
f = \cot x \quad u =
\]

\[
f' = \quad du =
\]

Option 2: Don't Limit Switch

Change back into \(x\)

delete example 3

\[
\int_0^1 \sin (3\pi x) \, dx
\]

\[
u = \quad d u =
\]
example 4

\[ \int \frac{1}{\sqrt{x} \left( \sqrt{x} + 1 \right)^2} \, dx \quad u = \]

\[ du = \]

example 5

\[ \int_{\frac{1}{2}}^{5} \frac{x}{\sqrt{x - 1}} \, dx \quad u = \Rightarrow \quad u = \]

\[ du = \quad x = 5 \Rightarrow u = \]

\[ x = 2 \Rightarrow u = \]
Integrals of Odd and Even Functions
Suppose that \( f \) is continuous on \([-a, a]\).

a. If \( f \) is even, then
\[
\int_{-a}^{a} f(x) \, dx = \int_{-a}^{a} (1 + x^2 - \cos x) \, dx =
\]

b. If \( f \) is odd, then
\[
\int_{-a}^{a} f(x) \, dx = \int_{-2}^{2} (x^3 - \sin x) \, dx =
\]

Area Between Curves
Consider the region \( S \) that lies between two curves
\[ y = f(x) \quad \text{and} \quad y = g(x) \]
and between the vertical lines
\[ x = a \quad \text{and} \quad x = b. \]

Here, \( f \) and \( g \) are continuous functions
and \( f(x) \geq g(x) \) for all \( x \) in \([a,b] \).
We divide $S$ into $n$ strips of equal width and approximate the $i$ th strip by a rectangle with base $\Delta x$ and height $f(x_i^*) - g(x_i^*)$.

The Riemann sum $\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$

is therefore an approximation to what we intuitively think of as the area of $S$.

This approximation appears to become better and better as $n \to \infty$.

Thus, we define the area $A$ of the region $S$ as the limiting value of the sum of the areas of these approximating rectangles.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$
Thus, we have the following formula for area:

\[ A = \int_a^b \left[ y = f(x) - y = g(x) \right] \, dx \]

Remember \( S \) is described as the region bounded by the curves \( y = f(x) \) and \( y = g(x) \) and the lines \( x = a \) and \( x = b \), where \( f \) and \( g \) are continuous and \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\).

Some regions are best treated by regarding \( x \) as a function of \( y \).

If a region is bounded by the curves \( x = f(y) \) and \( x = g(y) \) and the lines \( y = c \) and \( y = d \), where \( f \) and \( g \) are continuous and \( f(y) \geq g(y) \) for all \( y \) in \([c, d]\), then its area is:

\[ A = \int_c^d \left[ x = f(y) - x = g(y) \right] \, dy \]
Find the area of the region bounded by the curves.

\[ y = x^2 - 2x, \quad y = x + 4 \]

Select the correct answer:

a. \( \frac{125}{3} \)  b. \( \frac{25}{5} \)  c. 5  d. 20  e. \( \frac{125}{6} \)