Perhaps the most important of all the applications of calculus is to differential equations.

When physical or social scientists use calculus, more often than not, it is to analyze a differential equation that has arisen in the process of modeling some phenomenon they are studying.

It is often impossible to find an explicit formula for the solution of a differential equation.

- Nevertheless, we will see that graphical and numerical approaches provide the needed information.

In this section, we will learn:

- How to represent some mathematical models in the form of differential equations.

In describing the process of modeling in Section 1.2, we talked about formulating a mathematical model of a real-world problem through either:

- Intuitive reasoning about the phenomenon
- A physical law based on evidence from experiments
The model often takes the form of a differential equation.

- This is an equation that contains an unknown function and some of its derivatives.

This is not surprising.

- In a real-world problem, we often notice that changes occur, and we want to predict future behavior on the basis of how current values change.

Let’s begin by examining several examples of how differential equations arise when we model physical phenomena.

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population.

That is a reasonable assumption for a population of bacteria or animals under ideal conditions, such as:

- Unlimited environment
- Adequate nutrition
- Absence of predators
- Immunity from disease

Let’s identify and name the variables in this model:

- \( t \) = time (independent variable)
- \( P \) = the number of individuals in the population (dependent variable)
The rate of growth of the population is the derivative \( \frac{dP}{dt} \).

Hence, our assumption that the rate of growth of the population is proportional to the population size is written as the equation

\[
\frac{dP}{dt} = kP
\]

where \( k \) is the proportionality constant.

Equation 1 is our first model for population growth.

- It is a differential equation because it contains an unknown function \( P \) and its derivative \( \frac{dP}{dt} \).

Having formulated a model, let’s look at its consequences.

If we rule out a population of 0, then

\[ P(t) > 0 \text{ for all } t \]

So, if \( k > 0 \), then Equation 1 shows that:

\[ P'(t) > 0 \text{ for all } t \]

This means that the population is always increasing.

- In fact, as \( P(t) \) increases, Equation 1 shows that \( \frac{dP}{dt} \) becomes larger.
- In other words, the growth rate increases as the population increases.
Equation 1 asks us to find a function whose derivative is a constant multiple of itself.

- We know from Chapter 3 that exponential functions have that property.
- In fact, if we let \( P(t) = Ce^{kt} \), then \( P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t) \)

Thus, any exponential function of the form \( P(t) = Ce^{kt} \) is a solution of Equation 1.

In Section 9.4, we will see that there is no other solution.

Allowing \( C \) to vary through all the real numbers, we get the family of solutions \( P(t) = Ce^{kt} \), whose graphs are shown.

However, populations have only positive values.

- So, we are interested only in the solutions with \( C > 0 \).
- Also, we are probably concerned only with values of \( t \) greater than the initial time \( t = 0 \).

The figure shows the physically meaningful solutions.

Putting \( t = 0 \), we get:

\[ P(0) = Ce^{k(0)} = C \]

- The constant \( C \) turns out to be the initial population, \( P(0) \).
POPULATION GROWTH MODELS
Equation 1 is appropriate for modeling population growth under ideal conditions.

However, we have to recognize that a more realistic model must reflect the fact that a given environment has limited resources.

Many populations start by increasing in an exponential manner.

However, the population levels off when it approaches its carrying capacity $K$ (or decreases toward $K$ if it ever exceeds $K$).

POPULATION GROWTH MODELS
For a model to take into account both trends, we make two assumptions:
1. If $P$ is small.
   \[ \frac{dP}{dt} = kP \]
   (Initially, the growth rate is proportional to $P$.)

2. If $P > K$.
   \[ \frac{dP}{dt} < 0 \]
   ($P$ decreases if it ever exceeds $K$.)

A simple expression that incorporates both assumptions is given by the equation

\[ \frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \]

- If $P$ is small compared with $K$, then $P/K$ is close to 0. So, $dP/dt = kP$
- If $P > K$, then $1 - P/K$ is negative. So, $dP/dt < 0$

LOGISTIC DIFFERENTIAL EQUATION
Equation 2 is called the logistic differential equation.

- It was proposed by the Dutch mathematical biologist Pierre-François Verhulst in the 1840s—as a model for world population growth.

LOGISTIC DIFFERENTIAL EQUATIONS
In Section 9.4, we will develop techniques that enable us to find explicit solutions of the logistic equation.

- For now, we can deduce qualitative characteristics of the solutions directly from Equation 2.
We first observe that the constant functions $P(t) = 0$ and $P(t) = K$ are solutions.

- This is because, in either case, one of the factors on the right side of Equation 2 is zero.

This certainly makes physical sense. If the population is ever either 0 or at the carrying capacity, it stays that way.

- These two constant solutions are called equilibrium solutions.

If the initial population $P(0)$ lies between 0 and $K$, then the right side of Equation 2 is positive.

- So, $dP/dt > 0$ and the population increases.

However, if the population exceeds the carrying capacity ($P > K$), then $1 - P/K$ is negative.

- So, $dP/dt < 0$ and the population decreases.

Notice that, in either case, if the population approaches the carrying capacity ($P \to K$), then $dP/dt \to 0$.

- This means the population levels off.

So, we expect that the solutions of the logistic differential equation have graphs that look something like these.
Notice that the graphs move away from the equilibrium solution \( P = 0 \) and move toward the equilibrium solution \( P = K \).

Let’s now look at an example of a model from the physical sciences.

We consider the motion of an object with mass \( m \) at the end of a vertical spring.

In Section 6.4, we discussed Hooke’s Law.

- If the spring is stretched (or compressed) \( x \) units from its natural length, it exerts a force proportional to \( x \):

  \[
  \text{restoring force} = -kx
  \]

  where \( k \) is a positive constant (the spring constant).

If we ignore any external resisting forces (due to air resistance or friction) then, by Newton’s Second Law, we have:

\[
m \frac{d^2x}{dt^2} = -kx
\]
Let’s see what we can guess about the form of the solution directly from the equation.

We can rewrite Equation 3 in the form

\[
\frac{d^2 x}{dt^2} = -\frac{k}{m} x
\]

- This says that the second derivative of \( x \) is proportional to \( x \) but has the opposite sign.

We know two functions with this property, the sine and cosine functions.

- It turns out that all solutions of Equation 3 can be written as combinations of certain sine and cosine functions.

This is not surprising.

- We expect the spring to oscillate about its equilibrium position.
- So, it is natural to think that trigonometric functions are involved.

In general, a differential equation is an equation that contains an unknown function and one or more of its derivatives.

The order of a differential equation is the order of the highest derivative that occurs in the equation.

- Equations 1 and 2 are first-order equations.
- Equation 3 is a second-order equation.
In all three equations, the independent variable is called $t$ and represents time. However, in general, it doesn’t have to represent time.

For example, when we consider the differential equation

$$y' = xy$$

it is understood that $y$ is an unknown function of $x$.

A function $f$ is called a solution of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

- Thus, $f$ is a solution of Equation 4 if
  $$f(x) = xf(x)$$
  for all values of $x$ in some interval.

When we are asked to solve a differential equation, we are expected to find all possible solutions of the equation.

- We have already solved some particularly simple differential equations—namely, those of the form
  $$y' = f(x)$$

For instance, we know that the general solution of the differential equation $y' = x^3$ is given by

$$y = \frac{x^4}{4} + C$$

where $C$ is an arbitrary constant.

However, in general, solving a differential equation is not an easy matter.

- There is no systematic technique that enables us to solve all differential equations.
In Section 9.2, though, we will see how to draw rough graphs of solutions even when we have no explicit formula. We will also learn how to find numerical approximations to solutions.

**Example 1**

Show that every member of the family of functions

\[ y = \frac{1+ce^t}{1-ce^t} \]

is a solution of the differential equation

\[ \frac{1}{2} \left( y^2 - 1 \right) \]

The right side of the differential equation becomes:

\[
\frac{1}{2} \left( y^2 - 1 \right) = \frac{1}{2} \left( \frac{(1+ce^t)^2}{(1-ce^t)} - 1 \right) = \frac{1}{2} \left( \frac{(1+ce^t)^2 - (1-ce^t)^2}{(1-ce^t)^2} \right) = \frac{1}{2} \left( \frac{4ce^t}{(1-ce^t)^2} \right) = \frac{2ce^t}{(1-ce^t)^2}
\]

Therefore, for every value of \( c \), the given function is a solution of the differential equation.
SOLVING DIFFERENTIAL EQNS.

When applying differential equations, we are usually not as interested in finding a family of solutions (the general solution) as we are in finding a solution that satisfies some additional requirement.

- In many physical problems, we need to find the particular solution that satisfies a condition of the form \( y(t_0) = y_0 \).

INITIAL CONDITION & INITIAL-VALUE PROBLEM

This is called an initial condition.

The problem of finding a solution of the differential equation that satisfies the initial condition is called an initial-value problem.

INITIAL CONDITION

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point \((t_0, y_0)\).

INITIAL CONDITION

Physically, this corresponds to measuring the state of a system at time \( t_0 \) and using the solution of the initial-value problem to predict the future behavior of the system.

INITIAL CONDITION

Example 2

Find a solution of the differential equation

\[ y' = \frac{1}{2} (y^2 - 1) \]

that satisfies the initial condition \( y(0) = 2 \).

INITIAL CONDITION

Example 2

Substituting the values \( t = 0 \) and \( y = 2 \) into the formula from Example 1,

\[ y = \frac{1 + ce^t}{1 - ce^t} \]

we get:

\[ 2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c} \]
Solving this equation for $c$, we get:

$$2 - 2c = 1 + c$$

This gives $c = \frac{1}{3}$.

So, the solution of the initial-value problem is:

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}$$