We have looked at first-order differential equations from a geometric point of view (direction fields) and from a numerical point of view (Euler’s method).

- What about the symbolic point of view?

It would be nice to have an explicit formula for a solution of a differential equation.

- Unfortunately, that is not always possible.

In this section, we will learn about:

10.3 Separable Equations

In this section, we will learn about:

Certain differential equations that can be solved explicitly.

A separable equation is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of $x$ times a function of $y$.

- In other words, it can be written in the form $\frac{dy}{dx} = g(x)f(y)$.

The name separable comes from the fact that the expression on the right side can be “separated” into a function of $x$ and a function of $y$. 
Equivalently, if \( f(y) \neq 0 \), we could write

\[
\frac{dy}{dx} = \frac{g(x)}{h(y)}
\]

where \( h(y) = \frac{1}{f(y)} \).

To solve this equation, we rewrite it in the differential form

\[ h(y) \, dy = g(x) \, dx \]

so that:

- All \( y \)'s are on one side of the equation.
- All \( x \)'s are on the other side.

Then, we integrate both sides of the equation:

\[ \int h(y) \, dy = \int g(x) \, dx \]

Equation 2 defines \( y \) implicitly as a function of \( x \).

In some cases, we may be able to solve for \( y \) in terms of \( x \).

We use the Chain Rule to justify this procedure.

- If \( h \) and \( g \) satisfy Equation 2, then
  \[
  \frac{d}{dx} \left( \int h(y) \, dy \right) = \frac{d}{dx} \left( \int g(x) \, dx \right)
  \]

Thus,

\[
\frac{d}{dy} \left( \int h(y) \, dy \right) \frac{dy}{dx} = g(x)
\]

This gives:

\[ h(y) \frac{dy}{dx} = g(x) \]

Thus, Equation 1 is satisfied.
Example 1

a. Solve the differential equation
\[ \frac{dy}{dx} = \frac{x^2}{y^2} \]

b. Find the solution of this equation that satisfies the initial condition \( y(0) = 2 \).

Example 1 a

We write the equation in terms of differentials and integrate both sides:
\[ y^2 \, dy = x^2 \, dx \]
\[ \int y^2 \, dy = \int x^2 \, dx \]
\[ \frac{1}{3} y^3 = \frac{1}{3} x^3 + C \]
where \( C \) is an arbitrary constant.

Example 1 a

Solving for \( y \), we get:
\[ y = \sqrt[3]{x^3 + 3C} \]

We could leave the solution like this or we could write it in the form
\[ y = \sqrt[3]{x^3 + K} \]
where \( K = 3C \).

Since \( C \) is an arbitrary constant, so is \( K \).

Example 1 b

If we put \( x = 0 \) in the general solution in (a), we get:
\[ y(0) = \sqrt[3]{K} \]

To satisfy the initial condition \( y(0) = 2 \), we must have \( \sqrt[3]{K} = 2 \), and so \( K = 8 \).
So, the solution of the initial-value problem is:
\[ y = \sqrt[3]{x^3 + 8} \]

The figure shows graphs of several members of the family of solutions of the differential equation in Example 1.

- The solution of the initial-value problem in (b) is shown in red.
Solve the differential equation
\[ \frac{dy}{dx} = \frac{6x^2}{2y + \cos y} \]

**Example 2**

Writing the equation in differential form and integrating both sides, we have:
\[
(2y + \cos y) \ dy = 6x^2 \ dx
\]
\[
\int (2y + \cos y) \ dy = \int 6x^2 \ dx
\]
\[
y^2 + \sin y = 2x^3 + C
\]
where \( C \) is a constant.

**Example 2**

Equation 3 gives the general solution implicitly.

- In this case, it’s impossible to solve the equation to express \( y \) explicitly as a function of \( x \).

**Example 3**

Solve the equation
\[ y' = x^2y \]

- First, we rewrite the equation using Leibniz notation:
\[ \frac{dy}{dx} = x^2y \]

**Example 3**

If \( y \neq 0 \), we can rewrite it in differential notation and integrate:
\[ \frac{dy}{y} = x^2 \ dx \quad y \neq 0 \]
\[
\int \frac{dy}{y} = \int x^2 \ dx
\]
\[
\ln |y| = \frac{x^3}{3} + C
\]
The equation defines \( y \) implicitly as a function of \( x \).
However, in this case, we can solve explicitly for \( y \).

\[
|y| = e^{\ln|y|} = e^{\left(\frac{x^3}{3}\right) + C} = e^C e^{x^3/3}
\]

Hence,

\[
y = \pm e^C e^{x^3/3}
\]

We can easily verify that the function \( y = 0 \) is also a solution of the given differential equation.

- So, we can write the general solution in the form

\[
y = Ae^{x^3/3}
\]

where \( A \) is an arbitrary constant (\( A = e^C \), or \( A = -e^C \), or \( A = 0 \)).

The figure shows a direction field for the differential equation in Example 3.

If you use the direction field to sketch solution curves with \( y \)-intercepts 5, 2, 1, \(-1\), and \(-2\), they will resemble the curves in the figure.

In Section 9.2, we modeled the current \( I(t) \) in this electric circuit by the differential equation

\[
L \frac{dI}{dt} + RI = E(t)
\]

Find an expression for the current in a circuit where:

- The resistance is 12 \( \Omega \).
- The inductance is 4 H.
- A battery gives a constant voltage of 60 V.
- The switch is turned on when \( t = 0 \).

What is the limiting value of the current?
With \( L = 4 \), \( R = 12 \) and \( E(t) = 60 \),

- The equation becomes:
  \[
  4 \frac{dI}{dt} + 12I = 60 \quad \text{or} \quad \frac{dI}{dt} = 15 - 3I
  \]
- The initial-value problem is:
  \[
  \frac{dI}{dt} = 15 - 3I \quad I(0) = 0
  \]

We recognize this as being separable.

We solve it as follows:

\[
\int \frac{dI}{15 - 3I} = \int dt \quad (15 - 3I \neq 0)
\]

\[
- \frac{1}{3} \ln |15 - 3I| = t + C
\]

\[
|15 - 3I| = e^{3(t+C)}
\]

\[
15 - 3I = \pm e^{3(t+C)} = Ae^{-3t}
\]

\[
I = 5 - \frac{1}{3} Ae^{-3t}
\]

Since \( I(0) = 0 \), we have:

\[
5 - \frac{1}{3} A = 0
\]

So, \( A = 15 \) and the solution is:

\[
I(t) = 5 - 5e^{3t}
\]

The limiting current, in amperes, is:

\[
\lim_{t \to \infty} I(t) = \lim_{t \to \infty} (5 - 5e^{-3t})
\]

\[
= 5 - 5 \lim_{t \to \infty} e^{-3t}
\]

\[
= 5 - 0
\]

\[
= 5
\]

The figure shows how the solution in Example 4 (the current) approaches its limiting value.

Comparison with the other figure (from Section 9.2) shows that we were able to draw a fairly accurate solution curve from the direction field.
An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally—that is, at right angles.

Each member of the family \( y = mx \) of straight lines through the origin is an orthogonal trajectory of the family \( x^2 + y^2 = r^2 \) of concentric circles with center the origin.

- We say that the two families are orthogonal trajectories of each other.

Find the orthogonal trajectories of the family of curves \( x = ky^2 \), where \( k \) is an arbitrary constant.

The curves \( x = ky^2 \) form a family of parabolas whose axis of symmetry is the \( x \)-axis.

- The first step is to find a single differential equation that is satisfied by all members of the family.

To eliminate \( k \), we note that:

- From the equation of the given general parabola \( x = ky^2 \), we have \( k = x/y^2 \).
Hence, the differential equation can be written as:
\[
\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x^2}{y^2} y}
\]
or
\[
\frac{dy}{dx} = \frac{y}{2x}
\]
- This means that the slope of the tangent line at any point \((x, y)\) on one of the parabolas is: \(y' = y/(2x)\)

Orthogonal trajectories occur in various branches of physics.
- In an electrostatic field, the lines of force are orthogonal to the lines of constant potential.
- The streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.
If \( y(t) \) denotes the amount of substance in the tank at time \( t \), then \( y'(t) \) is the rate at which the substance is being added minus the rate at which it is being removed.

- The mathematical description of this situation often leads to a first-order separable differential equation.

We can use the same type of reasoning to model a variety of phenomena:

- Chemical reactions
- Discharge of pollutants into a lake
- Injection of a drug into the bloodstream

Example 6

A tank contains 20 kg of salt dissolved in 5000 L of water.

- Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min.
- The solution is kept thoroughly mixed and drains from the tank at the same rate.
- How much salt remains in the tank after half an hour?

Let \( y(t) \) be the amount of salt (in kilograms) after \( t \) minutes.

We are given that \( y(0) = 20 \) and we want to find \( y(30) \).

We do this by finding a differential equation satisfied by \( y(t) \).

Note that \( dy/dt \) is the rate of change of the amount of salt.

Thus,

\[
\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})
\]

where:

- ‘Rate in’ is the rate at which salt enters the tank.
- ‘Rate out’ is the rate at which it leaves the tank.

We have:

\[
\text{rate in} = \left(0.03 \frac{\text{kg}}{\text{L}}\right) \left(25 \frac{\text{L}}{\text{min}}\right)
\]

\[
= 0.75 \frac{\text{kg}}{\text{min}}
\]
The tank always contains 5000 L of liquid.

- So, the concentration at time $t$ is $y(t)/5000$ (measured in kg/L).

As the brine flows out at a rate of 25 L/min, we have:

\[
\text{rate out} = \left( \frac{y(t) \text{ kg}}{5000 \text{ L}} \right) \left( \frac{25 \text{ L}}{\text{min}} \right)
\]

\[
\frac{y(t) \text{ kg}}{200 \text{ min}}
\]

Thus, from Equation 5, we get:

\[
\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}
\]

- Solving this separable differential equation, we obtain:

\[
\int \frac{dy}{150 - y} = \int \frac{dt}{200} - \ln |150 - y| = \frac{t}{200} + C
\]

Since $y(0) = 20$, we have:

\[-\ln 130 = C\]

So,

\[-\ln |150 - y| = \frac{t}{200} - \ln 130\]

Therefore,

\[|150 - y| = 130e^{-t/200}\]

- $y(t)$ is continuous and $y(0) = 20$, and the right side is never 0.

- We deduce that $150 - y(t)$ is always positive.

Thus, $|150 - y| = 150 - y$.

So,

\[y(t) = 150 - 130e^{-t/200}\]

- The amount of salt after 30 min is:

\[y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}\]
Here’s the graph of the function $y(t)$ of Example 6.

- Notice that, as time goes by, the amount of salt approaches 150 kg.