12.2 Series

We will now add the terms of an infinite sequence \( \{a_n\}_{n=1}^{\infty} \) to get \( a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots \)
this is called an infinite _________.

Example:
\[ 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n-1} + \cdots \]

\( S_n \) = the sum of the first \( n \) terms
it is called the ___________
\[ S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \]

\( S_1 = \)
\( S_2 = 2 + \frac{2}{3} = \)
\( S_3 = 2 + \frac{2}{3} + \frac{2}{9} = \)
\( S_4 = 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} = \)

The partial sums form a sequence \( \{S_n\}_{n=1}^{\infty} \)
\[ \{S_n\}_{n=1}^{\infty} = \]

\[ \lim_{n \to \infty} S_n = s \quad \Rightarrow \text{We call } s \text{ the } _____ \text{ of the infinite series } \sum_{n=1}^{\infty} a_n = s \]
and the series is called ___________
(by adding sufficiently many terms of the series, we can get as close as we like to the number \( s \).)
otherwise the series is called ___________

The harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) ________.
We will show this in 12.3

\[ \{S_n\}_{n=1}^{\infty} = \left\{ 2, \frac{8}{3}, \frac{26}{9}, \frac{80}{27}, \cdots \right\} \]
It seems like \( \lim_{n \to \infty} S_n = 3 \)

\[ \Rightarrow 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots = \sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3 \]
We can show that the sum is 3 since this series is an example of a special type of series called a ________ series.
A geometric series is one in which each term is obtained from the preceding one by multiplying it by the common ratio \( r \).

\[
a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}
\]

this only converges \( \frac{a}{1-r} \).

\( r = 1 \)
\[
S_n = a + a + a + a + \cdots = na \Rightarrow \lim_{n \to \infty} na =
\]

\( r = -1 \)
\[
S_n = a - a + a - a + \cdots = (-1)^{n+1}a \Rightarrow \lim_{n \to \infty} (-1)^{n+1}a
\]

(\( \text{if it could be } a \), \( \text{it could be } 0 \))

\[
\text{depending on the value of } n
\]

\( r \neq \pm 1 \)

\[
S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a(1 + r + r^2 + r^3 + \cdots + r^{n-1})
\]

\[
rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a(r + r^2 + r^3 + \cdots + r^{n-1})
\]

\[
S_n - rS_n = a(1 - r^n)
\]

\[
\Rightarrow S_n (1 - r) = a(1 - r^n)
\]

\[
\Rightarrow S_n = \frac{a(1 - r^n)}{1 - r}
\]

so, \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a(1 - \lim_{n \to \infty} r^n)}{1 - r} = \frac{a}{1 - r} \)

We saw in section 12.1:

\[
\lim_{n \to \infty} r^n = \begin{cases} 
0 & \text{if } -1 < r < 1 \\
1 & \text{if } r = 1 \\
\text{undefined} & \text{if } r \in (-\infty, -1]
\end{cases}
\]

so, \( \lim_{n \to \infty} S_n = \frac{a}{1 - r} \) provided that \( -1 < r < 1 \) or \( |r| < 1 \).

The geometric series \( \sum_{n=1}^{\infty} ar^{n-1} \) converges to the sum of \( \frac{a}{1 - r} \) if \( |r| < 1 \)

The geometric series diverges for all other values of \( r \).
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Back to our example:

\[
2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots = 2 \left( \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n-1}} + \cdots \right)
\]

\(a = \) the first term
\(r = \) the ratio b/w the terms

Area of square = 1

\(sum\) of the series should also be 1

Find \(\sum_{n=1}^{\infty} a_n\).

\[
\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1
\]
Represent 2.15 as an improper fraction by using a geometric series.

\[ 2.15 = 2.151515\ldots \]

\[ = 2 + \]

a geometric series with \( a = \) _____ and \( r = \) _____

\[ s = \frac{a}{1 - r} = \]

A **telescoping series** is one in which the middle terms cancel and the sum collapses into just a few terms.

Example:

\[ \sum_{n=1}^{\infty} \frac{3}{n(n+3)} \]

\[ s_n = \]
If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then ____________.

Converse:
If $\lim_{n \to \infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent. This is _______

(Just because $\lim_{n \to \infty} a_n = 0$, you _______ conclude that the series $\sum_{n=1}^{\infty} a_n$ is convergent.)

Contrapositive:

Test for Divergence:

$$\sum_{n=1}^{\infty} \frac{3n^2}{n(n+3)}$$

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where $c$ is a constant),
$\sum (a_n + b_n)$, and $\sum (a_n - b_n)$,

i) $\sum ca_n =$

ii) $\sum (a_n + b_n) =$

iii) $\sum (a_n - b_n) =$