### 12.5 Alternating Series Test

An alternating series is of the form \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) or \( \sum_{n=1}^{\infty} (-1)^n b_n \), (where \( b_n > 0 \))

(it has successive terms of opposite signs)

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots
\]

Example:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n+5} = \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots
\]

Forms for the term that makes the series alternate in sign:

\[
(-1)^{n-1}, \quad (-1)^n, \quad (-1)^{n+1},
\]

\[
\cos(n\pi), \quad \sin\left(\frac{(2n-1)\pi}{2}\right)
\]

#### The Alternating Series Test

If the alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) (where \( b_n > 0 \)) satisfies:

i) \( \lim_{n \to \infty} b_n = 0 \)

ii) \( \{b_n\} \) is a decreasing sequence, and

then the series is **convergent**.

Note:

a) This test is for convergence only. It says nothing about divergence.

b) Like the function in the Integral Test, the sequence \( \{b_n\} \) needs to be decreasing "eventually" i.e., for all \( n > N \) for some \( N \)
Example 1:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad b_n = \frac{1}{n} \quad \lim_{n \to \infty} \frac{1}{n} = 0
\]

calculate the series convergent by the Alternating Series Test.

The Alternating Harmonic Series converges.

Example 2:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + 5} \quad b_n = \frac{n^2}{n^2 + 5} \quad \lim_{n \to \infty} \frac{n^2}{n^2 + 5} = 1
\]

the Alternating Series Test Does not apply.

\[
\lim a_n = \lim_{n \to \infty} \frac{(-1)^{n+1} n^2}{n^2 + 5} = \lim_{n \to \infty} (-1)^{n+1} \cdot \lim_{n \to \infty} \frac{n^2}{n^2 + 5} = (-1)^{n+1} \cdot 1 \Rightarrow \text{The limit does not exist.}
\]

The series diverges by the Test For Divergence, since does not exist.

Example 3:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n} \quad b_n = \frac{\ln n}{n}
\]

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\ln n}{n} = \frac{\ln \infty}{\infty} \quad \text{Indeterminate form} \Rightarrow \text{Use L'Hopitals Rule}
\]

\[
\lim_{n \to \infty} \frac{1}{n} = 0
\]

consider \( f(x) = \frac{\ln x}{x} \)

\[
f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}
\]

\[
f'(x) \text{ will be negative when } 1 - \ln x < 0 \quad \Rightarrow \ln x > 1 \quad e^{\ln x} > e^{1} \quad x > e
\]

\( \{b_n\} \text{ is decreasing for } n > 2 \)

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n} \text{ is convergent by the Alternating Series Test}
\]
### Alternating Series Estimation Theorem

If the alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) (where \( b_n > 0 \)) satisfies:

1. \( \lim_{n \to \infty} b_n = 0 \)
2. \( \{b_n\} \) is a decreasing sequence

then \( |R_n| = |s - s_n| \leq b_{n+1} \)

The size of the error is at most the size of the first omitted term.

The actual sum is between \( s_n - b_{n+1} \) and \( s_n + b_{n+1} \).

The error has the same sign as the first omitted term.

The error committed in using the 9th partial sum to approximate the total sum is \( R_9 \)

The size of this error is at most the size of the first omitted term.

\[
|R_9| = |s - s_9| \leq \frac{1}{100} \quad \Rightarrow \quad -\frac{1}{100} \leq s - s_9 \leq \frac{1}{100}
\]

\[
s_9 - \frac{1}{100} \leq s \leq s_9 + \frac{1}{100}
\]

The actual sum is between \( s_n - b_{n+1} \) and \( s_n + b_{n+1} \).

The sign of the error is the sign of the first omitted term.

\[
R_9 = s - s_9 < 0 \quad \Rightarrow \quad s_9 > s \quad \text{\textit{s}_9 \ \text{is an overestimate}}
\]

Since \( a_9 = -\frac{1}{100} \)