6.2 Volumes

**Goal:** To find the volume of a solid

**Method:** “Cutting” the solid into many “pieces”, find the volume of the pieces and add to find the total volume.

- The “pieces” are treated as cylinders.
- The base of each cylinder is called a **cross-section**.

The volume of each cylinder is found by taking the area of the cross-section, \( A(x_i^*) \), and multiplying by the height, \( \Delta x \).

- The volume of the solid can be approximated by the sum of all cylinders.

\[
V \approx \sum_{i=1}^{n} A(x_i^*) \Delta x
\]

- Taking the limit as the number of cylinders goes to infinity gives the exact volume of the solid.

\[
V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x \quad \Rightarrow V = \int_{a}^{b} A(x) \, dx
\]
A solid has a circular base of radius 4. If every plane cross section perpendicular to the $x$-axis is a square, then find the volume of the solid.

The equation of the circle is $x^2 + y^2 = 16$.

The integral is done in terms of $x$ since the squares are moved horizontally.

$x^2 + y^2 = 16 \implies y^2 = 16 - x^2 \implies y = \pm \sqrt{16 - x^2}$

The cross-sections are squares.

The length of the side of a square is $2\sqrt{16 - x^2}$.

The area of a cross-section is:

$$A(x) = \left(2\sqrt{16 - x^2}\right)^2 = A(x) = 4(16 - x^2)$$

$$V = \int_0^4 A(x) \, dx = \int_0^4 4(16 - x^2) \, dx = 2\int_0^4 (16 - x^2) \, dx$$

$$= 8\left(16x - \frac{x^3}{3}\right)_0^4 = 8\left(64 - \frac{64}{3}\right) = 8 \cdot 64 \left(1 - \frac{1}{3}\right)$$

$$= 8 \cdot 64 \left(\frac{2}{3}\right) = \frac{1024}{3}$$

When you revolve a plane region about an axis, the cross-sections are circular and the solid generated is called a **solid of revolution**.

If there is **no gap** between the axis of rotation and the region, then the method used is called the **disk method**.

If there is a **gap** between the axis of rotation and the region, then the method used is called the **washer method**.
Disk Method with horizontal axis of rotation (not necessarily the x-axis)

Cross-sections are circular: \[ A(x) = \pi \left( r(x) \right)^2 \]

radius as a function of \( x \)

Volume \[ \int_a^b A(x) \, dx = \int_a^b \pi \left( r(x) \right)^2 \, dx \]

radius as a function of \( x \)

Calculate the volume of the solid generated by rotating the region between the curves \( y = \sqrt{x} \) and \( y = 0 \) about the \( x \)-axis.

\[ A(x) = \pi \left[ \left( \sqrt{x} \right)^2 \right] = \pi x \]

\[ V = \int_0^a x \, dx = \pi \left[ \frac{x^2}{2} \right]_0^a = \frac{\pi}{2} a^3 \]
Calculate the volume of the solid generated by rotating the region between
the curves \( y = 4 - x^2 \) and \( y = 0 \) about the \( x \)-axis.

The radius in this case is the distance from the \( x \)-axis to the function

\[
r(x) = 4 - x^2
\]

Volume \( V = \int_a^b A(x)\,dx = \pi \int_a^b r(x)^2\,dx \)

\[
V = \pi \int_{-2}^{2} \left[4 - x^2\right]^2\,dx = \pi \int_{-2}^{2} \left[16 - 8x^2 + x^4\right]\,dx
\]

\[
= 2\pi \int_{0}^{2} \left[16 - 8x^2 + x^4\right]\,dx = 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2
\]

since the integrand is even

\[
= 2\pi \left[32 - \frac{32}{3} + \frac{32}{5}\right] = 2\pi \cdot 32 \left[1 - \frac{1}{3} + \frac{1}{5}\right] = 64\pi \left[15 - 10 + 3\right] \left[\frac{15}{15}\right]
\]

\[
= 64\pi \left[\frac{8}{15}\right] = \frac{512\pi}{15}
\]

Disk Method with vertical axis of rotation (not necessarily the \( y \)-axis)

Cross-sections are circular: \( A(y) = \pi \left[r(y)\right]^2 \)

Volume \( = \int_a^b A(y)\,dy = \pi \int_a^b \left[r(y)\right]^2\,dy \)
Calculate the volume of the solid generated by rotating the region between the curves \( y = x^3 \), \( y = 8 \), and \( x = 0 \) about the \( y \)-axis.

\[
V = \int_0^8 \pi y^2 \, dy - \int_0^2 \pi y^2 \, dy = \pi \left[ \frac{y^3}{3} \right]_0^8 - \pi \left[ \frac{y^3}{3} \right]_0^2 = \frac{532}{3} \pi
\]

Calculate the volume of the solid generated by rotating the region between the curves \( y = \frac{1}{x-2} \) and \( x = 2 \), \( y = \frac{1}{2} \), and \( y = 4 \) about the line \( x = 2 \).

The radius in this case is the horizontal distance from \( x = 2 \) to the curve \( y = \frac{1}{x-2} \).

We need to solve for \( x \) in terms of \( y \):

\[
y = \frac{1}{x-2} \Rightarrow \frac{y}{1} = \frac{1}{x-2} \Rightarrow y(x-2) = 1 \Rightarrow x-2 = \frac{1}{y} \Rightarrow x = \frac{1}{y} + 2
\]

The radius is then:

\[
r(y) = \frac{1}{y} + 2 \Rightarrow r(y) = \frac{1}{y}
\]

Volume:

\[
\int_\frac{1}{2}^4 A(y) \, dy = \int_\frac{1}{2}^4 \pi \left[ \frac{1}{y} \right]^2 \, dy = \pi \left[ \frac{1}{y} \right]_{\frac{1}{2}}^4 = \pi \left[ \frac{1}{4} - \frac{1}{\frac{1}{2}} \right] = \frac{7\pi}{4}
\]
Washer Method with **horizontal axis of rotation** (not necessarily the \(x\)-axis)

Draw a radius from the axis of rotation to the outer curve and call this **outer radius**

Draw a radius from the axis of rotation to the inner curve and call this **inner radius**

\[
Volume = \int_a^b A(x) \, dx = \pi \int_a^b \left( \left[ r_{\text{out}}(x) \right]^2 - \left[ r_{\text{in}}(x) \right]^2 \right) \, dx
\]

Calculate the volume of the solid generated by rotating the region between the curves \(y = 4 - x^2\) and \(y = 0\) about the line \(y = -2\).

The outside radius \(r_{\text{out}}\) can be found by drawing a line from the axis of rotation through the region.

The inside radius \(r_{\text{in}}\) can be found by drawing a line from the axis of rotation to the region.

\[
V = \pi \int_{-2}^2 \left( \left[ r_{\text{out}}(x) \right]^2 - \left[ r_{\text{in}}(x) \right]^2 \right) \, dx = \pi \int_{-2}^2 \left( 36 - 12x^2 + x^4 \right) \, dx
\]

\[
= \pi \left[ \left. 32x - 4x^3 + x^5 \right|_{-2}^2 \right] = 0
\]

\[
= 2\pi \left[ \int_0^2 \left( 32 - 12x^2 + x^4 \right) \, dx \right] = 2\pi \left[ \int_0^2 \left( 32x - 4x^3 + \frac{x^5}{5} \right) \, dx \right]
\]

\[
= 2\pi \left[ \left. 64 - 32 + \frac{32}{5} \right|_0^2 \right] = 2\pi \left[ 32 - 1 + \frac{1}{5} \right] = 64\pi \left[ \frac{6}{5} \right] = \frac{384\pi}{5}
\]
Washer Method with **vertical axis of rotation** (not necessarily the y-axis)

Draw a radius from the axis of rotation to the outer curve and call this **outer radius**

Draw a radius from the axis of rotation to the inner curve and call this **inner radius**

\[
Volume = \int_a^b A(y) \, dy = \pi \int_a^b \left( r_{\text{out}}(y)^2 - r_{\text{in}}(y)^2 \right) \, dy
\]

Calculate the volume of the solid generated by rotating the region between the curves \( y = \frac{x}{2} \) and \( y = \sqrt{x} \) about the \( y \)-axis.

We need to solve for \( x \) in terms of \( y \):

\[
\begin{align*}
    r_{\text{out}} &= 2y \\
    r_{\text{in}} &= y^2
\end{align*}
\]

\[
V = \pi \int_0^2 \left( (2y)^2 - y^2 \right) \, dy = \pi \int_0^2 \left( y^4 \right) \, dy = \pi \left( \frac{y^5}{5} \right) \bigg|_0^2 = \pi \left( \frac{32}{5} - \frac{32}{5} \right) = 32\pi \left( \frac{1}{3} - \frac{1}{5} \right)
\]

\[
= 32\pi \left( \frac{5 - 3}{15} \right) = \frac{64\pi}{15}
\]