9.3 Center of Mass 1-d

Archimedes' Law of the Lever

the rod will balance if

\[ m_1 d_1 = m_2 d_2 \]

moment of \( m_1 \) (with respect to the origin)

moment of \( m_2 \) (with respect to the origin)

moment of the system about the origin

\[ \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \]

total mass

If the total mass was concentrated at \( \bar{x} \), then its moment would be the same as the moment for the system.
9.3 Center of Mass 2-d

The center of mass is the point \( \left( \bar{x}, \bar{y} \right) \) where a single particle with the same mass as the total mass would have the same moments as the system.

\[ M_y = \bar{y} \text{ (total mass)} \]
\[ M_x = \bar{x} \text{ (total mass)} \]

\[ M_y = m_1 y_1 + m_2 y_2 + m_3 y_3 \]
\[ M_x = m_1 x_1 + m_2 x_2 + m_3 x_3 \]

\[ \bar{y} = \frac{M_y}{\text{total mass}} \]
\[ \bar{x} = \frac{M_x}{\text{total mass}} \]

9.3 Center of Mass 3-d

Consider a flat plate (called a lamina) with uniform density \( \rho \) that occupies a region \( \mathcal{R} \) of the plane.

The center of mass of the plate is called the **centroid** of \( \mathcal{R} \).
Thin plate: region between \( y = f(x) \) and \( y = g(x) \) with \( f(x) \geq g(x) \)

Constant density function \( \rho(x) = \rho \)

**Moment about the \( x \)-axis**

\[
M_x = \frac{\rho}{2} \int_a^b \left[ (f(x))^2 - (g(x))^2 \right] dx
\]

**Center of Mass**

\[
\begin{align*}
\bar{x} &= \frac{M_y}{M} = \frac{1}{A} \int_a^b x(f(x) - g(x)) dx \\
\bar{y} &= \frac{M_z}{M} = \frac{1}{2A} \int_a^b \left[ (f(x))^2 - (g(x))^2 \right] dx
\end{align*}
\]

Let \( A = \text{area} \) of the region b/w \( f(x) \) and \( g(x) \)

---

Thin plate: region between \( y = x - x^2 \) and \( y = -x \)

Constant density function \( \rho(x) = \rho \)

Limits of integration: \( x - x^2 = -y \)

\[
\begin{align*}
2x - x^2 &= 0 \\
x(2 - x) &= 0 \\
x = 0, \; x = 2
\end{align*}
\]

**Moment about the \( x \)-axis**

\[
\begin{align*}
M_x &= \frac{\rho}{2} \int_0^2 \left[ (x - x^2)^2 - (-x)^2 \right] dx \\
&= \frac{\rho}{2} \int_0^2 \left[ (x - x^2)^2 - (x^2)^2 \right] dx \\
&= \frac{\rho}{2} \int_0^2 \left[ (x - x^2)^2 + x^4 \right] dx \\
&= \frac{\rho}{2} \left[ \frac{-x^4}{2} + \frac{x^5}{5} \right]_0^2 = \frac{\rho}{2} \left[ \frac{-32}{5} \right] = \frac{\rho}{2} \left[ \frac{-40 + 32}{5} \right] = \frac{\rho}{2} \left[ \frac{-4}{5} \right] = \frac{-4}{5} \rho
\end{align*}
\]
Moment about the \( y-\text{axis} \)

\[
M_y = \rho \int_0^b x \cdot (f(x) - g(x)) \, dx
\]

\[
= \rho \int_0^2 x \left[ (x-x^2) - (-x) \right] \, dx = \rho \int_0^2 x \cdot (2x-x^2) \, dx = \rho \int_0^2 (2x^2-x^3) \, dx
\]

\[
= \rho \left[ \frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = \rho \left[ \frac{16}{3} - \frac{16}{4} \right] = \rho \left( \frac{16}{3} - 4 \right) = \rho \left( \frac{16-12}{3} \right) = \frac{4}{3} \rho
\]

Mass

\[
M = \rho \cdot \int_a^b \left( f(x) - g(x) \right) \, dx
\]

Area of the region b/w \( f(x) \) and \( g(x) \)

\[
= \rho \cdot \int_0^2 (2x-x^2) \, dx = \rho \left[ x^2 - \frac{x^3}{3} \right]_0^2 = \rho \left( 4 - \frac{8}{3} \right) = \rho \left( \frac{12-8}{3} \right) = \frac{4}{3} \rho
\]

Center of Mass

\[
\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}
\]

\[
\bar{x} = \frac{4}{3} \delta, \quad \bar{y} = \frac{-4}{3} \delta, \quad \bar{x} = 1, \quad \bar{y} = \frac{-3}{5}
\]

\[
(\bar{x}, \bar{y}) = \left( 1, \frac{-3}{5} \right)
\]
General Formulas

Thin plate: region under the graph of \( y = f(x) \) and above the \( x \)-axis

Constant density function \( \rho(x) = \rho \)

Set \( g(x) = 0 \) in the previous formulas.

\[
M_x = \frac{\rho}{2} \int_a^b [f(x)]^2 \, dx
\]

\[
M_y = \rho \int_a^b x f(x) \, dx
\]

\[
M = \rho \int_a^b f(x) \, dx
\]

Let \( A = \text{area under } f(x) \)

Center of Mass \( (\bar{x}, \bar{y}) \)

\[
\bar{x} = \frac{1}{A} \int_a^b x \cdot f(x) \, dx
\]

\[
\bar{y} = \frac{1}{2A} \int_a^b [f(x)]^2 \, dx
\]