FURTHER APPLICATIONS OF INTEGRATION

9.4 Applications to Economics and Biology

In this section, we will learn about:

Some applications of integration to economics and biology.

CONSUMER SURPLUS
Recall from Section 4.7 that the demand function $p(x)$ is the price a company has to charge in order to sell $x$ units of a commodity.

- Usually, selling larger quantities requires lowering prices.
- So, the demand function is a decreasing function.

DEMAND CURVE
The graph of a typical demand function, called a demand curve, is shown.

- If $X$ is the amount of the commodity that is currently available, then $P = P(X)$ is the current selling price.

CONSUMER SURPLUS
We divide the interval $[0, X]$ into $n$ subintervals, each of length $\Delta x = X/n$.

Then, we let $x_i^* = x_i$ be the right endpoint of the $i$th subinterval.

CONSUMER SURPLUS
Suppose, after the first $x_{i-1}$ units were sold, a total of only $x_i$ units had been available and the price per unit had been set at $p(x_i)$ dollars.

- Then, the additional $\Delta x$ units could have been sold (but no more).
CONSUMER SURPLUS
The consumers who would have paid $p(x_i)$ dollars placed a high value on the product.

- They would have paid what it was worth to them.

CONSUMER SURPLUS
Thus, in paying only $P$ dollars, they have saved an amount of:

\[(\text{savings per unit})(\text{number of units}) = [p(x_i) - P] \Delta x\]

CONSUMER SURPLUS
Taking similar groups of willing consumers for each of the subintervals and adding the savings, we get the total savings:

\[\sum_{i=1}^{n} [p(x_i) - P] \Delta x\]

CONSUMER SURPLUS
This sum corresponds to the area enclosed by the rectangles.

CONSUMER SURPLUS
If we let $n \to \infty$, this Riemann sum approaches the integral

\[\int_{0}^{X} [p(x) - P] dx\]

- Economists call this the consumer surplus for the commodity.

CONSUMER SURPLUS
The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price $P$, corresponding to an amount demanded of $X$. 

Definition 1
CONSUMER SURPLUS
The figure shows the interpretation of the consumer surplus as the area under the demand curve and above the line $p = P$.

CONSUMER SURPLUS
The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2$$

Find the consumer surplus when the sales level is 500.

Example 1

The number of products sold is $X = 500$.

So, the corresponding price is:

$$P = 1200 - (0.2)(500) - (0.0001)(500)^2$$
$$= 1075$$

Example 1

So, from Definition 1, the consumer surplus is:

$$\int_0^{500} [p(x) - P] dx$$
$$= \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075) dx$$
$$= \int_0^{500} (125 - 0.2x - 0.0001x^2) dx$$
$$= 125x - 0.1x^2 - 0.0001 \left( \frac{x^3}{3} \right) \bigg|_0^{500}$$
$$= (125)(500) - (0.1)(500)^2 - \frac{(0.0001)(500)^2}{3}$$
$$= $33,333.33$

BLOOD FLOW
In Example 7 in Section 3.7, we discussed the law of laminar flow:

$$v(r) = \frac{P}{4\eta l} (R^2 - r^2)$$

This gives the velocity $v$ of blood that flows along a blood vessel with radius $R$ and length $l$ at a distance $r$ from the central axis, where

- $P$ is the pressure difference between the ends of the vessel.
- $\eta$ is the viscosity of the blood.
Now, in order to compute the rate of blood flow, or flux (volume per unit time), we consider smaller, equally spaced radii \( r_1, r_2, \ldots \).

If \( \Delta r \) is small, then the velocity is almost constant throughout this ring and can be approximated by \( v(r_i) \).

Therefore, the volume of blood per unit time that flows across the ring is approximately

\[
(2\pi r_i \Delta r) v(r_i) = 2\pi r_i v(r_i) \Delta r
\]

The total volume of blood that flows across a cross-section per unit time is approximately

\[
\sum_{i=1}^{n} 2\pi r_i v(r_i) \Delta r
\]

Notice that the velocity (and hence the volume per unit time) increases toward the center of the blood vessel.

The approximation gets better as \( n \) increases.
FLUX
When we take the limit, we get the exact value of the flux (or discharge).

- This is the volume of blood that passes a cross-section per unit time.

BLOOD FLOW
\[ F = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi r_i v(r_i) \Delta r \]
\[ = \int_0^R 2\pi r v(r) \, dr \]
\[ = \int_0^R 2\pi r \frac{P}{4\eta l} (R^2 - r^2) \, dr \]
\[ = \frac{\pi P}{2\eta l} \int_0^R (R^2 r - r^4) \, dr \]
\[ = \frac{\pi P}{2\eta l} \left[ \frac{R^3}{2} - \frac{r^4}{4} \right]_{r=0}^{r=R} \]
\[ = \frac{\pi P}{2\eta l} \left( \frac{R^4}{2} - \frac{R^4}{4} \right) = \frac{\pi PR^4}{8\eta l} \]

POISEUILLE’S LAW
The resulting equation
\[ F = \frac{\pi PR^4}{8\eta l} \]
is called Poiseuille’s Law.

- It shows that the flux is proportional to the fourth power of the radius of the blood vessel.

CARDIAC OUTPUT
The figure shows the human cardiovascular system.

CARDIAC OUTPUT
Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs through the pulmonary arteries for oxygenation.

CARDIAC OUTPUT
Then, it flows back into the left atrium (through the pulmonary veins) and then out to the rest of the body (through the aorta).
CARDIAC OUTPUT
The cardiac output of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

DYE DILUTION METHOD
The dye dilution method is used to measure the cardiac output.

DYE DILUTION METHOD
Dye is injected into the right atrium and flows through the heart into the aorta.

DYE DILUTION METHOD
A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval $[0, T]$ until the dye has cleared.

DYE DILUTION METHOD
Let $c(t)$ be the concentration of the dye at time $t$.

- Let’s divide $[0, T]$ into subintervals of equal length $\Delta t$.
- Then, the amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is approximately
  
  \[
  \text{(concentration)(volume)} = c(t_i)F\Delta t
  \]
  
  where $F$ is the rate of flow we are trying to determine.

DYE DILUTION METHOD
Thus, the total amount of dye is approximately

\[
\sum_{i=1}^{n} c(t_i)F\Delta t = F\sum_{i=1}^{n} c(t_i)\Delta t
\]

Letting $n \to \infty$, we find that the amount of dye is:

\[
A = F \int_{0}^{T} c(t) \, dt
\]
DYE DILUTION METHOD

Thus, the cardiac output is given by

\[ F = \frac{A}{\int_0^T c(t) \, dt} \]

where the amount of dye \( A \) is known and the integral can be approximated from the concentration readings.

CARDIAC OUTPUT

A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals, as shown.

Estimate the cardiac output.

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<th>( t )</th>
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<tr>
<td>5</td>
<td>8.9</td>
<td></td>
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</tr>
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</table>

Example 2

Here, \( A = 5 \), \( \Delta t = 1 \), and \( T = 10 \).

We use Simpson’s Rule to approximate the integral of the concentration:

\[
\int_0^{10} c(t) \, dt = \frac{1}{3} [0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9) + 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0] = 41.87
\]

Example 2

Thus, Formula 3 gives the cardiac output to be:

\[
F = \frac{A}{\int_0^{10} c(t) \, dt} \approx \frac{5}{41.87} = 0.12 \text{L/s} = 7.2 \text{L/min}
\]