12.5 Equations of Lines and Planes

In order to find the equation of a line, we need:

A) a point on the line \( P_0(x_0, y_0, z_0) \)

B) a direction vector for the line \( \mathbf{v} = \langle a, b, c \rangle \)

\[ \mathbf{r} = \mathbf{r}_0 + t \mathbf{v} \quad \text{vector equation of line } L \]

\[ \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \]

equating components:

\[ x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \]

\( \text{parametric equations} \) of the line \( L \)

eliminating the parameter \( t \) (solve for \( t \) in each, then equate the results)

\[ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \]

\( \text{symmetric equations} \) of the line \( L \)

Find parametric equations of the line containing \((5,1,3)\) and \((3,-2,4)\).

In order to find the equation of a line, we need:

A) a point on the line \( P_0(x_0, y_0, z_0) \) \hspace{1cm} \text{Pick either point}

B) a direction vector for the line \( \mathbf{v} = \langle a, b, c \rangle \) \hspace{1cm} \text{Use the vector from one point to the other}

\[ L: \]

<table>
<thead>
<tr>
<th>Point</th>
<th>Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pick either point</td>
<td>( \mathbf{v} = \langle 3-5, -2-1, 4-3 \rangle )</td>
</tr>
<tr>
<td>( x ) = ( x_0 ) + ( at )</td>
<td>( \mathbf{v} = \langle -2, -3, 1 \rangle )</td>
</tr>
<tr>
<td>( y ) = ( y_0 ) + ( bt )</td>
<td></td>
</tr>
<tr>
<td>( z ) = ( z_0 ) + ( ct )</td>
<td></td>
</tr>
</tbody>
</table>

\[ L: \]

\[ x = 5 - 2t \]

\[ y = 1 - 3t \]

\[ z = 3 + t \]
Two lines in 3 space can interact in 3 ways:

A) **Parallel Lines** - their direction vectors are scalar multiples of each other

B) **Intersecting Lines** - there is a specific \( t \) and \( s \), so that the lines share the same point.

C) **Skew Lines** - their direction vectors are **not** parallel and there is **no** values of \( t \) and \( s \) that make the lines share the same point.

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Determine whether the lines \( L_1 \) and \( L_2 \) are parallel, skew or intersecting. If they intersect, find the point of intersection.

\[
L_1 \quad \begin{align*}
x &= 3 - t \\
y &= 5 + 3t \\
z &= -1 - 4t
\end{align*} \\
L_2 \quad \begin{align*}
x &= 8 + 2s \\
y &= -6 - 4s \\
z &= 5 + s
\end{align*}
\]

Set the \( x \)'s:

\[
\begin{align*}
3 - t &= 8 + 2s \\
-5 &= 5 + 2s \\
t &= -5 - 2s
\end{align*}
\]

Set the \( y \)'s:

\[
\begin{align*}
5 + 3t &= -6 - 4s \\
5 - 15 - 6s &= -6 - 4s \\
10 - 6s &= -2s \\
2s &= 4 \\
s &= -2
\end{align*}
\]

Check to make sure that the \( z \) values are equal for this \( t \) and \( s \).

\[
\begin{align*}
-1 - 4t &= 5 + s \\
-1 - 4(-1) &= 5 + (-2) \\
3 &= 3 \quad \text{check}
\end{align*}
\]

This should happen at the same time \( t \), so plug in \( t = -5 - 2s \)

\[
\begin{align*}
x &= 3 - (-1) \\
y &= 5 + 3(-1) \\
z &= -1 - 4(-1)
\end{align*}
\]

\[
\begin{align*}
L_1 \quad \text{using } L_1: \\
x &= 3 - (-1) \\
y &= 5 + 3(-1) \\
z &= -1 - 4(-1)
\end{align*}
\]

\( t = -5 - 2(-2) \)

\( t = 3 \)

\( t = -1 \)

Now find the pt. of intersection.

\[
\begin{align*}
x &= 3 - (-1) = (4, 2, 3) \\
y &= 5 + 3(-1) \\
z &= -1 - 4(-1)
\end{align*}
\]
Determine whether the lines $L_1$ and $L_2$ are parallel, skew or intersecting. If they intersect, find the point of intersection.

$L_1 \quad x = 4 + t \quad y = -8 - 2t \quad z = 12t$

$L_2 \quad x = 3 + 2s \quad y = -1 + s \quad z = -3 - 3s$

Set the $x$'s =

$4 + t = 3 + 2s$

$t = -1 + 2s$

Set the $y$'s =

$-8 - 2t = -1 + s$

This should happen at the same time $t$.

$-8 - 2(-1 + 2s) = -1 + s$

so plug in $t = -1 + 2s$

$-8 + 2 - 4s = -1 + s$

$t = -1 + 2s$

$-6 - 4s = -1 + s$

$s = -1$

$-5s = 5$

$5s = 5$

$t = -3$

Check to make sure that the $z$ values are equal for this $t$ and $s$.

$12(-3) = -3 - 3(-1)$

$-36 \neq 0$

This means that the $x$ and $y$ values are the same for $t = -3$ and $s = -1$, but the $z$ values are different.

The lines are skew

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**Planes**

In order to find the equation of a plane, we need:

A) a point on the plane $P_0(x_0, y_0, z_0)$

B) a vector that is orthogonal to the plane $n = \langle a, b, c \rangle$

this vector is called the normal vector to the plane

$$n \cdot (r - r_0) = 0$$

vector equation of the plane

$$n \cdot r = n \cdot r_0$$

Scalar equation of the plane

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

Linear equation of the plane

$$ax + by + cz + d = 0$$
Determine the equation of the plane that contains the lines $L_1$ and $L_2$.

$L_1: x = 3 - t \quad y = 5 + 3t \quad z = -1 - 4t$
$L_2: x = 8 + 2s \quad y = -6 - 4s \quad z = 5 + s$

In order to find the equation of a plane, we need:

A) a point on the plane
B) a vector that is orthogonal to the plane

**n = \langle a, b, c \rangle**

We have two points in the plane:
from $L_1: (3.5, -1)$ and from $L_2: (8, -6, 5)$

We have two vectors in the plane:
from $L_1: \langle 1, 3, 4 \rangle$ and from $L_2: \langle 2, 4, 1 \rangle$

Let $\mathbf{u} = \langle -1, 3, -4 \rangle$ and $\mathbf{v} = \langle 2, -4, 1 \rangle$

Find $\mathbf{u} \times \mathbf{v}$.

$$
\mathbf{u} \times \mathbf{v} = 
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 3 & -4 \\
2 & -4 & 1
\end{vmatrix}
= (3-16)\mathbf{i} + (-1)(-1+8)\mathbf{j} + (4-6)\mathbf{k}
= -13\mathbf{i} - 7\mathbf{j} - 2\mathbf{k} = \mathbf{n}
$$

$$
-13x - 7y - 2z + d = 0 \quad \text{or} \quad -39 - 35 + 2 + d = 0 \Rightarrow d = 72
$$

$$
13x + 7y + 2z - 72 = 0
$$

---

Determine the equation of the plane that passes through $(1, 2, 3), (3, 2, 1)$, and $(-1, -2, 2)$.

$PQ = \langle 3-1, 2-1, -3 \rangle = \langle 2, 1, -3 \rangle$
$PR = \langle -1-1, 2-2, 2-3 \rangle = \langle -2, 0, -1 \rangle$

Let $\mathbf{u} = \langle 2, 0, -2 \rangle$ and $\mathbf{v} = \langle -2, -4, -1 \rangle$

Find $\mathbf{u} \times \mathbf{v}$.

$$
\mathbf{u} \times \mathbf{v} = 
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 0 & -2 \\
-2 & -4 & -1
\end{vmatrix}
= (0-8)\mathbf{i} + (-1)(-2-4)\mathbf{j} + (8-0)\mathbf{k}
= -8\mathbf{i} - 6\mathbf{j} + 8\mathbf{k} = \mathbf{n} \text{ or n = 4i - 3j + 4k (in lowest terms)}
$$

$$
4x - 3y + 4z + d = 0
$$

$$
4(1) - 3(2) + 4(3) + d = 0 \Rightarrow 4 - 6 + 12 + d = 0 \Rightarrow d = -10
$$

$$
4x - 3y + 4z - 10 = 0
$$
Two distinct planes in 3-space either are **parallel** or **intersect in a line**.

Find the line of intersection of the two planes

\[
\begin{align*}
2x - 2y + z &= 0 \\
2x + 3y - 2z &= 0
\end{align*}
\]

Use the elimination method

\[
\begin{align*}
2x - 4y + 2z &= 0 \\
2x + 3y - 2z &= 0 \\
4x - y &= 0
\end{align*}
\]

\[
y = 4x
\]

Take the first equation:

\[
x - 2y + z = 0 \quad \text{and plug in} \quad y = 4x
\]

\[
x - 2(4x) + z = 0 \implies z = 7x
\]

\[x \text{ can be anything and we have } y \text{ and } z \text{ as functions of } x.\]

Let \( x = t \)

then \( y = 4t \) and \( z = 7t \)

\[
L: \\
x = t \\
y = 4t \\
z = 7t
\]
If two planes intersect, then you can determine the angle between them. 

\[ \angle \text{between planes} = \angle \text{between their normal vectors} \]

\[ \cos \theta = \left\| \mathbf{n}_1 \cdot \mathbf{n}_2 \right\| \]

Find the angle between the planes

\[ x - 2y + z = 0 \]
\[ 2x + 3y - 2z = 0 \]

\[ \Rightarrow \mathbf{n}_1 = \langle 1, -2, 1 \rangle \text{ and } \mathbf{n}_2 = \langle 2, 3, -2 \rangle \]

\[ |\mathbf{n}_1| = \sqrt{6} \text{ and } |\mathbf{n}_2| = \sqrt{17} \]

\[ \Rightarrow \mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 6 - 2 = -6 \]

**Distance between a point and a plane:**

1. Take any point in the plane call it \( P_0(x_0, y_0, z_0) \).
2. Let \( \mathbf{b} = \overrightarrow{P_0P} = \langle x_i - x_0, y_i - y_0, z_i - z_0 \rangle \)
3. Take \( \text{proj}_b \mathbf{b} \). The length of this vector = \( D \)
   
   Remember \( |\text{proj}_b \mathbf{b}| = \text{comp}_b \mathbf{b} = \frac{\mathbf{n} \cdot \mathbf{b}}{||\mathbf{n}||} \)
4. Since \( D \) must be positive, take the absolute value.

\[ D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{||\mathbf{n}||} \]

\[ \mathbf{n} \cdot \mathbf{b} = \langle a, b, c \rangle \cdot \langle x_i - x_0, y_i - y_0, z_i - z_0 \rangle = a(x_i - x_0) + b(y_i - y_0) + c(z_i - z_0) \]

\[ = ax_i + by_i + cz_i + \left( -ax_0 - by_0 - cz_0 \right) \]

\[ \Rightarrow |\mathbf{n} \cdot \mathbf{b}| = |ax_i + by_i + cz_i + d| \]

\[ D = \frac{|ax_i + by_i + cz_i + d|}{\sqrt{a^2 + b^2 + c^2}} \]
42. \((2, 2, 3), \ \ 2x + y + 2z = 4\)

\[
\begin{align*}
D &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \\
D &= \frac{2(2) + 2(2) + 3(3) - 4}{\sqrt{4+1+1}} = \frac{8}{3}
\end{align*}
\]
Distances
In Exercises 33–38, find the distance from the point to the line.

33. (0, 0, 12); \( x = 4t, \ y = -2t, \ z = 2t \)
34. (0, 0, 0); \( x = 5 + 3t, \ y = 5 + 4t, \ z = -3 - 5t \)
35. (2, 1, 3); \( x = 2 + 2t, \ y = 1 + 6t, \ z = 3 \)
36. (2, 1, -1); \( x = 2t, \ y = 1 + 2t, \ z = 2t \)
37. (3, -1, 4); \( x = 4 - t, \ y = 3 + 2t, \ z = -5 + 3t \)
38. (-1, 4, 3); \( x = 10 + 4t, \ y = -3, \ z = 4t \)

![Diagram showing points and vectors](image)