13.1 Curves in Space and their Tangents

VECTOR-VALUED FUNCTION

- vector function for short, sometimes called space curves
- takes one or more real variables and returns a vector
- we’ll focus on single variable ($t$) vector functions that
give three-dimensional vectors

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

deep valued functions called

component functions of $\mathbf{r}$

We want to look at the calculus of these vector functions.

Interesting vector valued functions:

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$$
called the circular helix

For a nice animation go to:
http://math.bu.edu/people/paul/225/circular_helix.html

$$\mathbf{r}(t) = \left\langle \left(2 + \cos\left(\frac{3t}{2}\right)\right)\cos t, \left(2 + \cos\left(\frac{3t}{2}\right)\right)\sin t, \sin\left(\frac{3t}{2}\right) \right\rangle$$
called the trefoil knot

For a nice animation go to:
http://math.bu.edu/people/paul/225/trefoil_knot.html
**Limit of Vector Functions**

\[ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \]

**Limit** of a vector function - taken component-wise

\[ \lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle \]

provided the component function limits exist

Let \( \mathbf{r}(t) = \left\langle \frac{\sin(t)}{t}, e^{2t}, \ln(1-t) \right\rangle \). Find \( \lim_{t \to 0} \mathbf{r}(t) \)

\[ \lim_{t \to 0} \mathbf{r}(t) = \left\langle \lim_{t \to 0} \frac{\sin(t)}{t}, \lim_{t \to 0} e^{2t}, \lim_{t \to 0} \ln(1-t) \right\rangle = \left\langle 1, 1, 0 \right\rangle \]

\[ \lim_{t \to 0} \mathbf{r}(t) = \langle 1, 1, 0 \rangle = \mathbf{i} + \mathbf{j} \]

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**Differentiation of Vector Functions**

\[ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \]

**Derivative** of a vector function - taken component-wise

\[ \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle \]

provided the component function are differentiable

Let \( \mathbf{r}(t) = \left\langle \sin^{-1}t, \sqrt{1-t^2}, \ln(1+3t) \right\rangle \). Find \( \mathbf{r}'(t) \)

\[ \mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1-t^2}}, \frac{1}{2}(1-t^2)^{-1/2}(-2t), \frac{1}{1+3t} \right\rangle \]

\[ \mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1-t^2}}, \frac{-t}{\sqrt{1-t^2}}, \frac{3}{1+3t} \right\rangle \]
**Particle Motion**

\( \mathbf{r}(t) \) is the position vector
\( \mathbf{r}'(t) \) is the tangent vector
\( \mathbf{r}'(t) \) will now be called the **velocity vector**, \( \mathbf{v}(t) \)
\( |\mathbf{r}'(t)| \) is called the **speed** of the particle

\( \mathbf{r}''(t) \) is called the **acceleration vector**, \( \mathbf{a}(t) \)

\[
\mathbf{r}(t) = \langle \sin t, 2 \cos t \rangle \\
\mathbf{v}(t) = \mathbf{r}'(t) = \langle \cos t, -2 \sin t \rangle \\
\mathbf{a}(t) = \mathbf{r}''(t) = \langle -\sin t, -2 \cos t \rangle \\
|\mathbf{r}'(t)| = \sqrt{\cos^2 t + 4 \sin^2 t} \\
\mathbf{r}\left(\frac{\pi}{6}\right) = \langle \frac{1}{2}, \sqrt{3} \rangle \\
\mathbf{v}\left(\frac{\pi}{6}\right) = \langle \frac{\sqrt{3}}{2}, -1 \rangle \\
\mathbf{a}\left(\frac{\pi}{6}\right) = \langle -\frac{1}{2}, -\sqrt{3} \rangle
\]

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**Differentiation Rules:**

1. \( \frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t) \)
2. \( \frac{d}{dt} [c\mathbf{u}(t)] = c\mathbf{u}'(t) \)
3. \( \frac{d}{dt} [f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \)
4. \( \frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \)
5. \( \frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \)
6. \( \frac{d}{dt} [\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)) \)
Show that if $|\mathbf{r}(t)| = c$ (constant), then $\mathbf{r}'(t) \perp \mathbf{r}(t)$ for all $t$.

Assume $|\mathbf{r}(t)| = c$.

The fact that $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$, gives us $\mathbf{r}(t) \cdot \mathbf{r}(t) = c^2$.

Taking the derivative of both sides gives $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = 0$.

Using the properties of differentiation, we get $\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

The dot product is commutative, so $\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$

or $2[\mathbf{r}'(t) \cdot \mathbf{r}(t)] = 0$.

So, $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$ and thus $\mathbf{r}'(t) \perp \mathbf{r}(t)$ (for all $t$)

Most vector functions do not have constant magnitude.

The helix example from earlier: $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$|\mathbf{r}(t)| = \sqrt{\cos^2(t) + \sin^2(t) + t^2}$

$|\mathbf{r}(t)| = \sqrt{1 + t^2}$

It’s magnitude isn’t constant (it depends on $t$).

Consider its derivative however: $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$|\mathbf{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$

$\mathbf{r}'(t)$ has constant magnitude, so $\mathbf{r}'(t) \perp \mathbf{r}'(t)$

So for the helix, the velocity vector will always be orthogonal to the acceleration vector (don’t think that this always happens)
Show that if \( \mathbf{r}(t) \) is a vector function such that \( \mathbf{r}''(t) \) exists, then

\[
\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)
\]

Using the properties of differentiation, we get

\[
\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t)
\]

anytime you cross a vector

with itself, you get the zero vector

So, \( \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) \).

When taking the derivative of the cross product of a vector function and its velocity, you can just take the cross product of the vector function and its acceleration.
The **tangent line** to a smooth curve

\[ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad \text{at } t = t_0 \]

passes through the point \( (f(t_0), g(t_0), h(t_0)) \),

and has the same direction as the velocity vector at \( t_0 \)

\[ \langle f'(t_0), g'(t_0), h'(t_0) \rangle. \]

So the parametric equations of the tangent line are:

\[
\begin{align*}
  x &= f(t_0) + f'(t_0)t \\
  y &= g(t_0) + g'(t_0)t \\
  z &= h(t_0) + h'(t_0)t 
\end{align*}
\]