14.5 Directional Derivatives and Gradients

\[ z = f(x, y) \]

\[ f_x(x_0, y_0) \] is the rate of change of \( z \) in the direction parallel to the \( x \)-axis.

\[ f_y(x_0, y_0) \] is the rate of change of \( z \) in the direction parallel to the \( y \)-axis.

What about the rate of change of \( z \) in other directions?

Want to find the rate of change of \( z \) at \( (x_0, y_0, z_0) \) in the direction of an arbitrary unit vector \( \mathbf{u} = \langle a, b \rangle \).

Surface \( S \) with equation \( z = f(x, y) \)

The vertical plane that passes through \( P \) in the direction of \( \mathbf{u} \) intersects the \( S \) in a curve \( C \).

The slope of the tangent line \( T \) to \( C \) at the point \( P \) is the rate of change of \( z \) in the direction of \( \mathbf{u} \).

Let \( Q(x, y, z) \) be another point on \( C \).

Project \( P \) and \( Q \) onto the \( xy \)-plane to get \( P' \) and \( Q' \).

\[ \overrightarrow{PQ} = \mathbf{u} = \langle ha, hb \rangle \]

\[ ha = x - x_0 \] and \( hb = y - y_0 \)

\[ x = x_0 + ha \] and \( y = y_0 + hb \)

\[ z = f(x, y) = f(x_0 + ha, y_0 + hb) \]
The directional derivative of \( f(x, y) \) in the direction of \( \mathbf{u} \) at \( (x_0, y_0) \) is given by:

\[
D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
\]

It is more practical to find \( D_{\mathbf{u}} f \) by using the gradient.

Gradient of a function of two variables \( z = f(x, y) \):

\[
\nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)
\]

If you want to know the rate of change of \( f \) in the direction of an angle \( \theta \) with the positive \( x \)-axis, then \( \mathbf{u} = (\cos \theta, \sin \theta) \).

\[
D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot (\cos \theta, \sin \theta)
\]
Find the directional derivative of \( f(x, y) = x^2 - 3xy + 4y^3 \) at the point \( P(-2, 0) \) in the direction of \( a = i + 2j \)

\[
D_uf(-2, 0) = \nabla f(-2, 0) \cdot u
\]

\[
f(x, y) = x^2 - 3xy + 4y^3 \Rightarrow \nabla f(x, y) = \langle 2x - 3y, -3x + 12y^2 \rangle
\]

\[
\nabla f(-2, 0) = \langle -4, 6 \rangle
\]

\[
a = i + 2j \Rightarrow |a| = \sqrt{5} \Rightarrow u = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle
\]

\[
D_uf(-2, 0) = \langle -4, 6 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{-4}{\sqrt{5}} + \frac{12}{\sqrt{5}} = \frac{8}{\sqrt{5}}
\]

Find the directional derivative of \( f(x, y, z) = \frac{z-x}{z+y} \) at the point \( P(1,0,-3) \) in the direction of \( a = -6i + 3j - 2k \)

\[
D_uf(1,0,-3) = \nabla f(1,0,-3) \cdot u
\]

\[
f(x, y, z) = \frac{z-x}{z+y} \quad f_x(x, y, z) = -\frac{1}{z+y} \quad f_y(x, y, z) = \frac{y-x}{(z+y)^2} \quad f_z(x, y, z) = \frac{x-y}{(z+y)^2}
\]

\[
\nabla f(x, y, z) = \left\langle \frac{-1}{z+y}, \frac{x-z}{(z+y)^2}, \frac{y-x}{(z+y)^2} \right\rangle \Rightarrow \nabla f(1,0,-3) = \left\langle \frac{1}{3}, \frac{4}{9}, \frac{1}{9} \right\rangle
\]

\[
a = -6i + 3j - 2k \Rightarrow |a| = \sqrt{36 + 9 + 4} = \sqrt{49} = 7 \Rightarrow u = \left\langle \frac{-6}{7}, \frac{3}{7}, \frac{-2}{7} \right\rangle
\]

\[
D_uf(1,0,-3) = \left\langle \frac{1}{3}, \frac{4}{9}, \frac{1}{9} \right\rangle \cdot \left\langle \frac{-6}{7}, \frac{3}{7}, \frac{-2}{7} \right\rangle = \frac{1}{3} \left( \frac{-6}{7} \right) + \frac{4}{9} \left( \frac{3}{7} \right) + \frac{1}{9} \left( \frac{-2}{7} \right)
\]

\[
= -\frac{18 + 12 - 2}{63} = -\frac{8}{63}
\]
The maximum value of the directional derivative at \((x_0, y_0)\) is \(\|\nabla f(x_0, y_0)\|\)
and occurs when \(\mathbf{u}\) has the same direction as \(\nabla f(x_0, y_0)\).

\[
D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = \|\nabla f(x_0, y_0)\| \cdot |\mathbf{u}| \cos \theta \\
= \|\nabla f(x_0, y_0)\| \cos \theta \quad \text{since} \quad |\mathbf{u}| = 1
\]

will be maximized

when \(\cos \theta = 1\)

\(\Rightarrow\) The maximum value of \(D_u f(x_0, y_0)\) is \(\|\nabla f(x_0, y_0)\|\)

and \(\theta = 0 \Rightarrow \mathbf{u}\) has the same direction as \(\nabla f(x_0, y_0)\).

The minimum value of the directional derivative at \((x_0, y_0)\) is \(-\|\nabla f(x_0, y_0)\|\)

and occurs when \(\mathbf{u}\) has the opposite direction as \(\nabla f(x_0, y_0)\).

\(\nabla f(x_0, y_0)\) is perpendicular to the level curve \(f(x, y) = k\) that passes through the point \(P(x_0, y_0)\)

On a topographical map, if \(f(x, y)\) represents the height above sea level
at a point with coordinates \((x, y)\), the path of steepest ascent is perpendicular
to all the contour lines.
Let $S$ be a surface with equation $F(x, y, z) = k$

- $S$ is a level surface of a function $F$ of three variables

Let $P(x_0, y_0, z_0)$ be a point on $S$.

Let $C$ be any curve that lies on the surface $S$ and passes through the point $P$.

Let $C$ defined by $r(t) = \langle x(t), y(t), z(t) \rangle$

Any point $\langle x(t), y(t), z(t) \rangle$ on $C$ is also on $S$.

$\Rightarrow F(x(t), y(t), z(t)) = k$

If $x, y,$ and $z$ are differentiable functions of $t$ and $F$ is also differentiable, then

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

Another way to write this is:

$$\nabla F \cdot r'(t) = 0$$

$\Rightarrow \nabla F \perp r'(t)$

Let $t_0$ be the parameter value corresponding to $P$.

$$r(t_0) = \langle x_0, y_0, z_0 \rangle$$

The tangent vector $r'(t_0)$ lies in the tangent plane to the surface $S$ at the point $P$.

$$\nabla F \cdot r'(t_0) = 0 \Rightarrow \nabla F(x_0, y_0, z_0)$$

is the normal vector to the tangent plane to $S$ at $P$

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The equation of the plane with normal $\langle a, b, c \rangle$ containing the point $\langle x_0, y_0, z_0 \rangle$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

When the plane is the tangent plane to the surface $F(x, y, z) = k$ at the point $\langle x_0, y_0, z_0 \rangle$:

$$a = F_x(x_0, y_0, z_0) \quad b = F_y(x_0, y_0, z_0) \quad c = F_z(x_0, y_0, z_0)$$

$$0(x - x_0) + F_x(x_0, y_0, z_0)(y - y_0) + F_y(x_0, y_0, z_0)(z - z_0) = 0$$

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**Spring 2006 Final**

5. Let $S$ be the surface $x^2 + 4z^2 - yz = 0$. An equation for the tangent plane to $S$ at $(1, 2, -1)$ is

$$F_x = 2x + 4z \quad F_y = 0 \quad F_z = 12xz^2 - y$$

$$F_x(1, -2, 1) = 4 \quad F_y(1, -2, 1) = 1 \quad F_z(1, -2, 1) = 12 - 2$$

$0(x - 1) + 2(y - 2) + 10(z + 1) = 0$

$2y - 4 + 10z + 10 = 0 \Rightarrow 2y + 10z + 6 = 0$

$y + 5z + 3 = 0$