A function of two variables has a **local maximum** at \((a, b)\) if \(f(x, y) \leq f(a, b)\) for all points \((x, y)\) in some region around \((a, b)\).

- outside the region it is possible that the function could be larger

A function of two variables has a **local minimum** at \((a, b)\) if \(f(x, y) \geq f(a, b)\) for all points \((x, y)\) in some region around \((a, b)\).

- outside the region it is possible that the function could be smaller

A point \((a, b)\) is called a **critical point** of \(f\) if one of the following is true:

1. \(\nabla f(a, b) = 0\), that is BOTH \(f_x(a, b) = 0\) and \(f_y(a, b) = 0\)

2. \(f_x(a, b)\) and / or \(f_y(a, b)\) doesn't exist

\(f\) has a local maximum or local minimum at \((a, b)\) and the first partial derivatives of \(f\) exist there

\[ f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0 \]

\((a, b)\) is a critical point

\(f\) has a local maximum or local minimum at \((a, b)\)

\((a, b)\) is a critical point

not all critical points lead to a local maximum or local minimum
Find all critical points \((a, b)\) such that \(f_x(a, b) = 0\) and \(f_y(a, b) = 0\) and the second partial derivatives are continuous in some region around \((a, b)\)

Let

\[
D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (f_{xx})(f_{yy}) - (f_{xy})^2
\]

Evaluate \(D\) at these critical points

- \(D(a, b) > 0\) \(\Rightarrow\) \((a, b)\) is a local minimum
- \(D(a, b) < 0\) \(\Rightarrow\) \((a, b)\) is a local maximum
- \(D(a, b) = 0\)\(\Rightarrow\) the test gives no information

\(f(0,0) = 0\)

Maximum in the direction of the x-axis

Minimum in the direction of the y-axis

The graph is in the shape of a saddle.
The point \((0,0,0)\) is called a saddle point.
\[ f(x, y) = 3x^2 y + y^3 - 3x^2 - 3y^2 + 2 \]
\[ f_x = 6xy - 6x \quad f_y = 3x^2 + 3y^2 - 6y \]
\[ f_x = 6x(y - 1) \]
\[ f_x = 0 \Rightarrow \text{either } 6x = 0 \text{ or } y - 1 = 0 \]
\[ (a) \quad x = 0 \Rightarrow f_y = 3y^2 - 6y = 3y(y - 2) \Rightarrow y = 0 \text{ or } y = 2 \]
\[ (0,0) \text{ and } (0.2) \]
\[ (b) \quad y = 1 \Rightarrow f_y = 3x^2 + 3 - 6 = 3(x^2 - 1) \Rightarrow x = 1 \text{ or } x = -1 \]
\[ (1,1) \text{ and } (-1,1) \]

\[ f(x, y) = 3x^2 y + y^3 - 3x^2 - 3y^2 + 2 \]
\[ f_x = 6xy - 6x \quad f_y = 3x^2 + 3y^2 - 6y \]
\[ f_{xx} = 6y - 6 \quad f_{yy} = 6y - 6 \quad \text{ } D(0,0) = 36 \quad D(0,2) = 36 \]
\[ f_{xx}(0,0) = -6 \quad f_{xx}(0,2) = 6 \]
\[ f_{xy} = 6x \]
\[ D = (6y - 6)^2 - (6x)^2 \]
\[ D(1,1) = -36 \quad D(-1,1) = -36 \]
\[ f_{xx}(1,1) = \text{doesn't matter} \quad f_{xx}(-1,1) = \text{doesn't matter} \]

\[ D \quad f_{xx} \quad \text{Classification} \]
\[ \begin{array}{ccc}
(0,0,2) & >0 & < 0 \quad \text{local max.} \\
(0.2,-2) & >0 & > 0 \quad \text{local min.} \\
(1.1,0) & <0 & -- \quad \text{saddle pt.} \\
(-1.1,0) & <0 & -- \quad \text{saddle pt.}
\end{array} \]
A function of two variables has an **absolute maximum** at \((a, b)\) if \(f(x, y) \leq f(a, b)\) for all points in the domain of \(f\).

A function of two variables has an **absolute minimum** at \((a, b)\) if \(f(x, y) \geq f(a, b)\) for all points in the domain of \(f\).

Usually the domain is restricted to some region

\[f(x, y) = x^2 + y^2 + x^2 y + 4\]

Restricted Domain:

\[-1 \leq x \leq 1\] and \[-1 \leq y \leq 1\]

A region in \(\mathbb{R}^2\) (for us this will be the \(xy\) plane) is called **closed** if it includes its boundary.

A region in \(\mathbb{R}^2\) (for us this will be the \(xy\) plane) is called **bounded** if it is contained within some disk (in other words a region is bounded if it is finite)

**Extreme Value Theorem**

\(f(x, y)\) is continuous in some closed bounded region \(S\) in \(\mathbb{R}^2\) \(\Rightarrow\) there are points \((a, b)\) and \((c, d)\) in the region \(S\) so that \(f(a, b)\) is an absolute maximum and \(f(c, d)\) is an absolute minimum.

This tells us that the points exist but it doesn’t tell us how to find them.
To find the absolute maximum and absolute minimum values of a continuous function \( f \) on a closed region \( S \):

1) Find all the critical points of \( f \) that lie in the region \( S \)
   Evaluate the function at each of these points
2) Find all extreme values of \( f \) that lie on the boundary. (This turns into a Calc I problem)
3) Find all the "corner points" of the region
   and evaluate the function at these points.
4) The largest and smallest of the function values found in steps 1-3 are the absolute maximum value and absolute minimum value of the function \( f \)

Find the absolute maximum and absolute minimum values of \( f(x, y) = x^2 + xy \) on the region \( S : \{(x, y) ||x| \leq 2, |y| \leq 1\} \)

**Find critical points** inside the region and on the boundary

\[
\begin{align*}
  f_x &= 2x + y = 0 \\
  f_y &= x = 0 \\
  \Rightarrow & x = y = 0 \quad (0, 0)
\end{align*}
\]

\( L_1 : x = -2 \)

\[
\begin{align*}
  f(x, y) &\text{ on } L_1 : 4 - 2y \\
  g &= 4 - 2y \Rightarrow g' = -2 \text{ never 0.}
\end{align*}
\]

\( L_2 : y = 1 \)

\[
\begin{align*}
  f(x, y) &\text{ on } L_2 : x^2 + x \\
  g &= x^2 + x \Rightarrow g' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}
\end{align*}
\]

\( L_3 : x = 2 \)

\[
\begin{align*}
  f(x, y) &\text{ on } L_3 : 4 + 2y \\
  g &= 4 + 2y \Rightarrow g' = 2 \text{ never 0.}
\end{align*}
\]

\( L_4 : y = -1 \)

\[
\begin{align*}
  f(x, y) &\text{ on } L_4 : x^2 - x \\
  g &= x^2 - x \Rightarrow g' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}
\end{align*}
\]
Find the absolute maximum and absolute minimum values of \( f(x, y) = x^2 + xy \) on the region \( S : \{(x, y) ||x| \leq 2, |y| \leq 1\} \)

\[
\begin{align*}
  f(0, 0) = 0 & \quad \text{abs. max.} \\
  f\left(\frac{1}{4}, 1\right) = \frac{5}{4} & \quad \text{abs. min.} \\
  f\left(\frac{1}{4}, -1\right) = -\frac{5}{4}
\end{align*}
\]

Corner points:
\[
\begin{align*}
  f(2, 1) = 6 & \quad f(-2, -1) = 0 \\
  f(-2, -1) = 6 & \quad f(2, 1) = 0 \\
  f(2, -1) = 2 & \quad f(-2, 1) = 2
\end{align*}
\]