Section 16.2 Vector Fields

Velocity vector fields showing the wind speed and direction

(a) 12:00 AM, February 20, 2007
(b) 2:00 PM, February 21, 2007

(a) Ocean currents off the coast of Nova Scotia
DEFINITION Let \( D \) be a set in \( \mathbb{R}^2 \) (a plane region). A vector field on \( \mathbb{R}^2 \) is a function \( F \) that assigns to each point \((x, y)\) in \( D \) a two-dimensional vector \( F(x, y) \).

\[
F(x, y) = \langle -y, x \rangle
\]
16.1/16.2 Vector Fields and Line Integrals

\( \mathbf{F}(x, y) = \langle y, \sin x \rangle \)

\( \mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + y^2) \rangle \)

A) \( \mathbf{F}(x, y) = \langle -y, x \rangle \)

B) \( \mathbf{F}(x, y) = \langle 1, \sin y \rangle \)

C) \( \mathbf{F}(x, y) = \langle x - 2, x + 1 \rangle \)

D) \( \mathbf{F}(x, y) = \left\langle y, \frac{1}{x} \right\rangle \)
**Definition** Let \( E \) be a subset of \( \mathbb{R}^3 \). A vector field on \( \mathbb{R}^3 \) is a function \( \mathbf{F} \) that assigns to each point \((x, y, z)\) in \( E \) a three-dimensional vector \( \mathbf{F}(x, y, z) \).

\[
\mathbf{F}(x, y, z) = \langle y, z, x \rangle
\]

\[
\mathbf{F}(x, y, z) = \langle y, -2, x \rangle
\]

\[
\mathbf{F}(x, y, z) = \left\langle \frac{y}{z}, -\frac{x}{z}, \frac{z}{4} \right\rangle
\]

**Physics applications**

**Velocity Field**

- Dynamics
- Fluid dynamics
- Aerodynamics
- Thermodynamics

**Force Field**

- Gravitational Field
- Electric Field
- Magnetic Field
Gradient Vector Field

\[ f(x, y) \]
\[ \nabla f(x, y) = \{ f_x(x, y), f_y(x, y) \} \]

\[ f(x, y) = x^2 y - y^3 \quad \nabla f(x, y) = \{2xy, x^2 - 3y^2\} \]

As we saw in 15.6, the gradient vectors are perpendicular to the level curves.

Gradient vectors are long where the level curves are close and short where they are far apart.

Directional derivative

maximum value

\[ |\nabla f| \]

A vector field \( \mathbf{F} \) is called \textbf{conservative} if it is the gradient of some scalar function.

If there exists a function \( f \) such that \( \mathbf{F} = \nabla f \), then \( \mathbf{F} \) is conservative.

\( f \) is called the \textbf{potential function} for \( \mathbf{F} \).

\[ f(x, y) = x^2 y - y^3 \quad \mathbf{F} = \{2xy, x^2 - 3y^2\} \]

In 17.3 we learn how to tell whether a vector field is conservative and how to find \( f \) when it is.
Section 16.1 Line Integrals

Parametric Curve
\[ x = f(t), \quad y = g(t) \]

- (a) Smooth curve
- (b) Piecewise-smooth curve
- (d) Simple closed curve
- (c) Closed but not simple

Consists of a finite number of smooth curves
Starts and ends at the same point and doesn't cross itself
Starts and ends at the same pt.

Orientation of the curve

Geometric Interpretation of a Line Integral

Definite Integral

Line Integral

(a) Vertical rectangle

(b) "Fence" or "curtain" of varying height \( G(x, y) \) with base \( C \)
\[ \int_C P(x, y) \, dx + Q(x, y) \, dy = \int_a^b U(t) f'(t) \, dt + V(t) g'(t) \, dt \]

Parametrize the path
\[ x = f(t), \quad y = g(t), \quad a \leq t \leq b \]
\[ dx = f'(t) \, dt, \quad dy = g'(t) \, dt \]
Substitute everything
\[ P(x, y) = P(f(t), g(t)) = U(t) \]
\[ Q(x, y) = Q(f(t), g(t)) = V(t) \]

Work done by a vector field \( \mathbf{F} \) as its point of application moves along \( C \) from \( A \) to \( B \).

Calc II
\[ C \rightarrow \text{line} \]
\[ W = \mathbf{F} \cdot d \]

Now
\[ C \rightarrow \text{curve} \]
\( r \) the parametrization of \( C \) as a vector
\[ W = \int_C \mathbf{F} \cdot d\mathbf{r} \]

Evaluate \( \int_C y \, dx + x \, dy \)
on \( C \) : line segments from \((0, 0)\) to \((1, 0)\)and from \((1, 0)\) to \((1, 1)\)
\[ \int_C y \, dx + x \, dy = \int_{C_1} y \, dx + x \, dy + \int_{C_2} y \, dx + x \, dy \]
\[ C_1: \quad x = t, \quad y = 0, \quad 0 \leq t \leq 1 \]
\[ dx = dt, \quad dy = 0 \]
\[ \int_{C_1} y \, dx + x \, dy = \int_0^1 (0 + 0) \, dt = 0 \]
\[ C_2: \quad x = 1, \quad y = t, \quad 0 \leq t \leq 1 \]
\[ dx = 0, \quad dy = dt \]
\[ \int_{C_2} y \, dx + x \, dy = \int_0^1 (0 + 1) \, dt = 1 \]
\[ \int_C y \, dx + x \, dy = \boxed{1} \]
Evaluate \( \int_C (x^2 + y^2) \, dx - 2xy \, dy \) on the given closed curve \( C \).

\[ \int_C (x^2 + y^2) \, dx - 2xy \, dy = \int_{C_1} (x^2 + y^2) \, dx - 2xy \, dy + \int_{C_2} (x^2 + y^2) \, dx - 2xy \, dy \]

\( C_1 \):
\[
\begin{align*}
&x = t \\
y = t^2 \\
dx = dt \\
dy = 2tdt \\
&0 \leq t \leq 1
\end{align*}
\]

\[
\int_{C_1} (x^2 + y^2) \, dx - 2xy \, dy = \int_0^1 (t^2 + t^4 - 4t^4) \, dt = \int_0^1 (t^2 - 3t^4) \, dt
\]

\[
= \left[ \frac{t^3}{3} - \frac{3t^5}{5} \right]_0^1 = \frac{1}{3} - \frac{3}{5} = \frac{-9}{15} = -\frac{3}{5}
\]

\( C_2 \):
\[
\begin{align*}
&x = t \\
y = \sqrt{t} \\
dx = dt \\
dy = \frac{1}{2\sqrt{t}} \, dt \\
&t \text{ starts at } 1 \text{ and ends at } 0
\end{align*}
\]

\[
\int_{C_2} (x^2 + y^2) \, dx - 2xy \, dy = \int_1^0 (t^2 + t - t) \, dt = -\int_0^1 t^2 \, dt = -\frac{1}{3}
\]

\[
\int_C (x^2 + y^2) \, dx - 2xy \, dy = \frac{-4}{15} + \frac{1}{3} = \frac{-4 - 5}{15} = -\frac{9}{15} = -\frac{3}{5}
\]
Find the work done by the force \( \mathbf{F}(x, y) = 2xy \mathbf{i} + 4y^2 \mathbf{j} \) acting along the piecewise smooth curve consisting of line segments from \((-2, 2)\) to \((0, 0)\) and from \((0, 0)\) to \((2, 3)\).

\[
\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]

\( C_1: \) \[
\begin{align*}
x &= t \\
y &= -t \\
\frac{dx}{dt} &= 1 \\
\frac{dy}{dt} &= -1 \\
2 \leq t \leq 0
\end{align*}
\]

\( r = \mathbf{a} - t\mathbf{j} \Rightarrow d\mathbf{r} = \langle 1, -1 \rangle dt \)

\( \mathbf{F} \) on \( C_1: \) \( \langle -2t^2, 4t^3 \rangle \)

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^{0} \left( -2t^2 - 4t^3 \right) dt = -2 \left[ t^2 ight]_{-2}^{0} - 4 \left[ t^3 \right]_{-2}^{0} = -2(0 - (-8)) = -16
\]

\( C_2: \) \[
\begin{align*}
x &= t \\
y &= \frac{3}{2}t \\
\frac{dx}{dt} &= 1 \\
\frac{dy}{dt} &= \frac{3}{2}
\end{align*}
\]

\( 0 \leq t \leq 2 \)

\( r = \mathbf{a} + \frac{3}{2}t\mathbf{j} \Rightarrow d\mathbf{r} = \left\langle 1, \frac{3}{2} \right\rangle dt \)

\( \mathbf{F} \) on \( C_2: \) \( \langle 3t^2, 9t^3 \rangle \)

\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2} \left( 3t^2 + \frac{27}{2}t^3 \right) dt = \int_{0}^{2} \left( \frac{33}{2}t^2 \right) dt = \frac{33}{2} \left[ t^3 \right]_{0}^{2} = \frac{33}{2} (8 - 0) = 44
\]

\[
\text{Work} = \int_{C} \mathbf{F} \cdot d\mathbf{r} = -16 + 44 = 28
\]