PARTIAL DERIVATIVES

15.4

Tangent Planes and Linear Approximations

In this section, we will learn how to:

Approximate functions using tangent planes and linear functions.

TANGENT PLANES

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives.

Let $P(x_0, y_0, z_0)$ be a point on S.

TANGENT PLANES **Equation 2**

Suppose f has continuous partial derivatives.

An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

TANGENT PLANES **Example 1**

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

- Let $f(x, y) = 2x^2 + y^2$.
- Then,

$f_x(x, y) = 4x$ $\quad f_y(x, y) = 2y$

$f_x(1, 1) = 4$ $\quad f_y(1, 1) = 2$

TANGENT PLANES **Example 1**

- So, Equation 2 gives the equation of the tangent plane at (1, 1, 3) as:

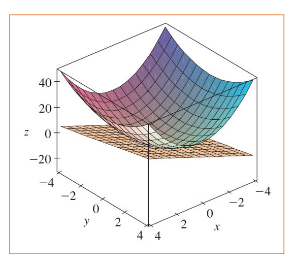
$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

TANGENT PLANES

The figure shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1.



LINEAR APPROXIMATIONS

In Example 1, we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is:

$$z = 4x + 2y - 3$$

LINEAR APPROXIMATIONS

Thus, in view of the visual evidence in the previous two figures, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1).

LINEARIZATION & LINEAR APPROXIMATION

The function L is called the linearization of f at (1, 1).

The approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the linear approximation or tangent plane approximation of f at (1, 1).

LINEAR APPROXIMATIONS

For instance, at the point (1.1, 0.95), the linear approximation gives:

$$f(1.1, 0.95)$$
$$\approx 4(1.1) + 2(0.95) - 3$$
$$= 3.3$$

- This is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$

LINEAR APPROXIMATIONS

However, if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation.

- In fact, $L(2, 3) = 11$, whereas $f(2, 3) = 17$.

LINEAR APPROXIMATIONS

In general, we know from Equation 2 that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

LINEARIZATION **Equation 3**

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linearization of f at (a, b).

LINEAR APPROXIMATION **Equation 4**

The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the tangent plane approximation of f at (a, b).

LINEAR APPROXIMATIONS **Theorem 8**

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

LINEAR APPROXIMATIONS **Example 2**

Show that $f(x, y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there.

Then, use it to approximate $f(1.1, -0.1)$.

LINEAR APPROXIMATIONS **Example 2**

The partial derivatives are:

$f_x(x, y) = e^{xy} + xye^{xy}$ $\quad$ $f_y(x, y) = x^2e^{xy}$

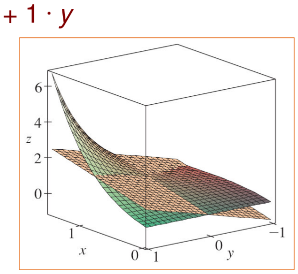
$f_x(1, 0) = 1$ $\quad$ $f_y(1, 0) = 1$

- Both f_x and f_y are continuous functions.
- So, f is differentiable by Theorem 8.

LINEAR APPROXIMATIONS **Example 2**

The linearization is:

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$
$$= 1 + 1(x - 1) + 1 \cdot y$$
$$= x + y$$



LINEAR APPROXIMATIONS **Example 2**

The corresponding linear approximation is:

$$xe^{xy} \approx x + y$$

So,

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

- Compare this with the actual value of

$$f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$$

DIFFERENTIALS

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable.

- That is, dx can be given the value of any real number.

DIFFERENTIALS **Equation 9**

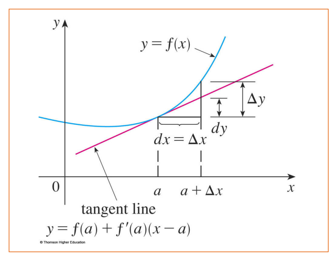
Then, the differential of y is defined as:

$$dy = f'(x)\, dx$$

- See Section 3.10

DIFFERENTIALS

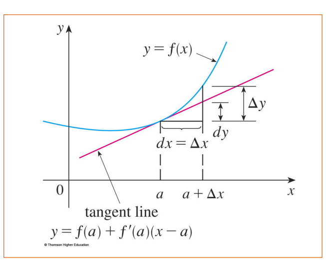
The figure shows the relationship between the increment Δy and the differential dy.



DIFFERENTIALS

Δy represents the change in height of the curve $y = f(x)$.

dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.



DIFFERENTIALS

For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables.

- That is, they can be given any values.

TOTAL DIFFERENTIAL **Equation 10**

Then the differential *dz*, also called the total differential, is defined by:

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

- Compare with Equation 9.
- Sometimes, the notation *df* is used in place of *dz*.

DIFFERENTIALS

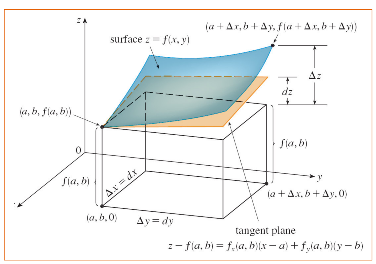
If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of *z* is:

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- So, in the notation of differentials, the linear approximation in Equation 4 can be written as:
  $f(x, y) \approx f(a, b) + dz$

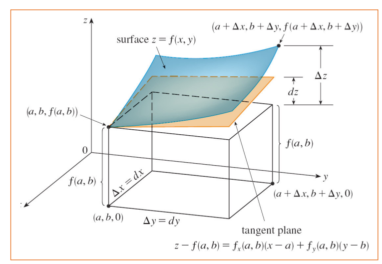
DIFFERENTIALS

The figure is the three-dimensional counterpart of the previous figure.



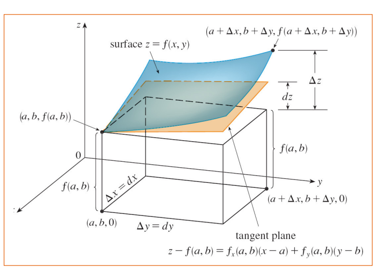
DIFFERENTIALS

It shows the geometric interpretation of the differential *dz* and the increment Δ*z*.



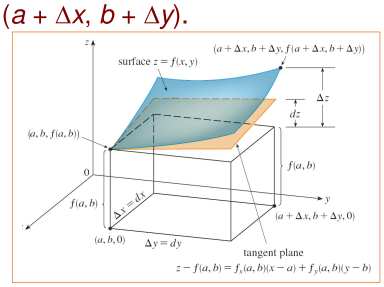
DIFFERENTIALS

dz is the change in height of the tangent plane.



DIFFERENTIALS

Δ*z* represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.



DIFFERENTIALS **Example 4**

a. If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

b. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare Δz and dz.

DIFFERENTIALS **Example 4 a**

Definition 10 gives:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
$$= (2x + 3y)\,dx + (3x - 2y)\,dy$$

DIFFERENTIALS **Example 4 b**

Putting

$x = 2$, $dx = \Delta x = 0.05$, $y = 3$, $dy = \Delta y = -0.04$,

we get:

$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04)$

$= 0.65$

DIFFERENTIALS **Example 4 b**

The increment of z is:

$\Delta z = f(2.05, 2.96) - f(2, 3)$

$= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2]$

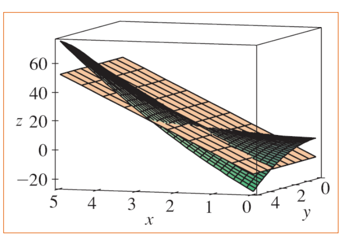
$- [2^2 + 3(2)(3) - 3^2]$

$= 0.6449$

- Notice that $\Delta z \approx dz$, but dz is easier to compute.

DIFFERENTIALS

In Example 4, dz is close to Δz because the tangent plane is a good approximation to the surface $z = x^2 + 3xy - y^2$ near (2, 3, 13).



DIFFERENTIALS **Example 5**

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each.

- Use differentials to estimate the maximum error in the calculated volume of the cone.

DIFFERENTIALS **Example 5**

The volume V of a cone with base radius r and height h is $V = \pi r^2 h/3$.

So, the differential of V is:

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh$$

DIFFERENTIALS **Example 5**

Each error is at most 0.1 cm.

So, we have:

$|\Delta r| \leq 0.1$

$|\Delta h| \leq 0.1$

DIFFERENTIALS **Example 5**

To find the largest error in the volume, we take the largest error in the measurement of r and of h.

- Therefore, we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$.

DIFFERENTIALS **Example 5**

That gives:

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1)$$
$$= 20\pi$$

- So, the maximum error in the calculated volume is about 20π cm^3 ≈ 63 cm^3.

FUNCTIONS OF THREE OR MORE VARIABLES

The differential dw is defined in terms of the differentials dx, dy, and dz of the independent variables by:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

MULTIPLE VARIABLE FUNCTIONS **Example 6**

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm.

- Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

MULTIPLE VARIABLE FUNCTIONS **Example 6**

If the dimensions of the box are x, y, and z, its volume is $V = xyz$.

Thus,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$
$$= yz\, dx + xz\, dy + xy\, dz$$

MULTIPLE VARIABLE FUNCTIONS **Example 6**

We are given that

$$|\Delta x| \le 0.2,\ |\Delta y| \le 0.2,\ |\Delta z| \le 0.2$$

- To find the largest error in the volume, we use
  $dx = 0.2,\ dy = 0.2,\ dz = 0.2$
  together with
  $x = 75,\ y = 60,\ z = 40$

MULTIPLE VARIABLE FUNCTIONS **Example 6**

Thus,

$$\Delta V \approx dV$$
$$= (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2)$$
$$= 1980$$

MULTIPLE VARIABLE FUNCTIONS **Example 6**

So, an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3 in the calculated volume.

- This may seem like a large error.
- However, it's only about 1% of the volume of the box.