A function of two variables has a **local maximum** at \((a, b)\) if \(f(x, y) \leq f(a, b)\) for all points \((x, y)\) in some region around \((a, b)\).

- outside the region it is possible that the function could be larger

A function of two variables has a **local minimum** at \((a, b)\) if \(f(x, y) \geq f(a, b)\) for all points \((x, y)\) in some region around \((a, b)\).

- outside the region it is possible that the function could be smaller

A point \((a, b)\) is called a **critical point** of \(f\) if one of the following is true:

1) \(\nabla f(a, b) = 0\), that is BOTH \(f_x(a, b) = 0\) and \(f_y(a, b) = 0\)

2) \(f_x(a, b)\) and \(f_y(a, b)\) doesn't exist

\(f\) has a local maximum or local minimum at \((a, b)\) and the first partial derivatives of \(f\) exist there

\[
\begin{align*}
\nabla f(a, b) &= 0 \\
\Rightarrow & \quad f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0 \\
\downarrow & \quad (a, b) \text{ is a critical point}
\end{align*}
\]

\(f\) has a local maximum or local minimum at \((a, b)\) not all critical points lead to a local maximum or local minimum

\[
\begin{align*}
\nabla f(a, b) &= 0 \\
\Rightarrow & \quad (a, b) \text{ is a critical point}
\end{align*}
\]
Given: \( z = \frac{1}{2} (y^2 - x^2) \)

- Maximum in the direction of the x-axis
- Minimum in the direction of the y-axis

The point \((0,0,0)\) is called a **saddlepoint**.

To find all critical points \((a,b)\) such that \( f_x(a,b) = 0 \) and \( f_y(a,b) = 0 \), and the second partial derivatives are continuous in some region around \((a,b)\), we use the

\[
D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (f_{xx})(f_{yy}) - (f_{xy})^2
\]

Evaluate \( D \) at these critical points.

- If \( D(a,b) > 0 \) and \( f_{xx}(a,b) > 0 \), \( f(a,b) \) is a **local minimum**.
- If \( D(a,b) > 0 \) and \( f_{xx}(a,b) < 0 \), \( f(a,b) \) is a **local maximum**.
- If \( D(a,b) < 0 \), \( f(a,b) \) is a **saddle point**.
- If \( D(a,b) = 0 \), the test gives no information.
\[ f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2 \]
\[ f_x = 6xy - 6x \quad f_y = 3x^2 + 3y^2 - 6y \]
\[ f_x = 6x(y-1) \]
\[ f_x = 0 \Rightarrow \text{either } 6x = 0 \text{ or } y-1 = 0 \]
\[ (a) \quad x = 0 \Rightarrow f_y = 3y^2 - 6y = 3y(y-2) \]
\[ \Rightarrow y = 0 \text{ or } y = 2 \]
\[ (0,0) \text{ and } (0,2) \]
\[ (b) \quad y = 1 \Rightarrow f_y = 3x^2 + 3 - 6 = 3(x^2 - 1) \]
\[ \Rightarrow x = 1 \text{ or } x = -1 \]
\[ (1,1) \text{ and } (-1,1) \]

\[ D = (6y - 6)^2 - (6x)^2 \]
\[ D = 36([y-1]^2 - x^2) \]
\[ f_{xx}(0,0) = -6 \quad f_{xx}(0,2) = 12 \]
\[ D(0,0) = 36 \quad D(0,2) = 36 \]

\[ f_{xx}(1,1) = \text{doesn't matter} \]
\[ f_{xx}(-1,1) = \text{doesn't matter} \]
\[ D(1,1) = -36 \quad D(-1,1) = -36 \]

\[ \begin{array}{c|c|c}
D & f_{xx} & \text{Classification} \\
\hline
(0,0,2) & >0 & <0 \text{ local max.} \\
(0,2,-2) & >0 & >0 \text{ local min.} \\
(1,1,0) & <0 & - - \text{ saddle pt.} \\
(-1,1,0) & <0 & - - \text{ saddle pt.} \\
\end{array} \]
A function of two variables has an **absolute maximum** at \((a, b)\) if \(f(x, y) \leq f(a, b)\) for all points in the domain of \(f\).

A function of two variables has an **absolute minimum** at \((a, b)\) if \(f(x, y) \geq f(a, b)\) for all points in the domain of \(f\).

Usually the domain is restricted to some region

\[ f(x, y) = x^2 + y^2 + x^2y + 4 \]

Restricted Domain: 

\[-1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\]

A region in \(\mathbb{R}^2\) (for us this will be the \(xy\) plane) is called **closed** if it includes its boundary.

A region in \(\mathbb{R}^2\) (for us this will be the \(xy\) plane) is called **bounded** if it is contained within some disk (in other words a region is bounded if it is finite).

**Extreme Value Theorem**

\(f(x, y)\) is continuous in some closed bounded region \(S\) in \(\mathbb{R}^2\)

\[\Rightarrow\]

there are points \((a, b)\) and \((c, d)\) in the region \(S\) so that \(f(a, b)\) is an absolute maximum and \(f(c, d)\) is an absolute minimum

This tells us that the points exist but it doesn’t tell us how to find them.
To find the absolute maximum and absolute minimum values of a continuous function $f$ on a closed region $S$:

1) Find all the critical points of $f$ that lie in the region $S$
   Evaluate the function at each of these points

2) Find all extreme values of $f$ that lie on
   the boundary. (This turns into a Calc I problem)

3) The largest and smallest of the values found in
   steps 1 and 2 are the absolute maximum value and absolute
   minimum value of the function $f$

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Find the absolute maximum and absolute minimum values of
$f(x, y) = x^2 + xy$ on the region $S : \{(x, y) | |x| \leq 2, |y| \leq 1\}$

1) Find possible critical pts. inside the region
   
   $f(x, y) = x^2 + xy$
   
   $f_x = 2x + y = 0$
   
   $f_y = x = 0$
   
   Both have to be true at the same time
   
   plugging in $x = 0$ into $f_y \Rightarrow y = 0$

2) Find all extreme values of $f$ that lie on the boundary.
   
   a) $L_4 : x = -2$
   
   $f(-2, y) = (-2)^2 + (-2)y$ \(\Rightarrow f\) on $L_4 = -2y + 4$

   b) $L_2 : y = 1$

   $f(x, 1) = x^2 + x$

   \(\Rightarrow f\) on $L_2 = x^2 + x$

   \(f'\) on $L_2 = 2x + 1 = 0$ \(\Rightarrow x = \frac{-1}{2}\)

   \(\left(\frac{-1}{2}, 1\right)\)

   no extreme points on $L_2$

   c) $L_3 : x = 2$

   $f(2, y) = 2^2 + 2y$

   \(\Rightarrow f\) on $L_3 = 2y + 4$

   \(f'\) on $L_3 = 2 \neq 0$

   no extreme points on $L_3$

   d) $L_1 : y = -1$

   $f(x, -1) = x^2 - x$

   \(\Rightarrow f\) on $L_1 = x^2 - x$

   \(f'\) on $L_1 = 2x - 1 = 0$ \(\Rightarrow x = \frac{1}{2}\)

   $\left(\frac{1}{2}, -1\right)$

$f(0, 0) = 0$

\(f\left(\frac{-1}{2}, 1\right) = \frac{-1}{2}\)

\(f\left(\frac{1}{2}, -1\right) = \frac{1}{2}\)

absolute maximum

absolute minimum

absolute minimum