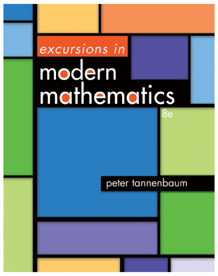


Chapter 13: Fibonacci Numbers and the Golden Ratio

13.1 Fibonacci Numbers

THE FIBONACCI SEQUENCE

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

The sequence of numbers shown above is called the **Fibonacci sequence**, and the individual numbers in the sequence are known as the **Fibonacci numbers**.

Fibonacci Sequence

You should recognize these numbers as the number of pairs of rabbits in Fibonacci's rabbit problem as we counted them from one month to the next.

The Fibonacci sequence is infinite, and except for the first two 1s, each number in the sequence is the sum of the two numbers before it.

Fibonacci Number

We will denote each Fibonacci number by using the letter *F* (for Fibonacci) and a *subscript* that indicates the *position* of the number in the sequence. In other words, the first Fibonacci number is $F_1 = 1$, the second Fibonacci number is $F_2 = 1$, the third Fibonacci number is $F_3 = 2$, the tenth Fibonacci number is $F_{10} = 55$. We may not know (yet) the numerical value of the 100th Fibonacci number, but at least we can describe it as F_{100}.

Fibonacci Number

A generic Fibonacci number is usually written as F_N (where *N* represents a generic position). If we want to describe the Fibonacci number that comes before F_N we write F_{N-1} ; the Fibonacci number two places before F_N is F_{N-2}, and so on. Clearly, this notation allows us to describe relations among the Fibonacci numbers in a clear and concise way that would be hard to match by just using words.

Fibonacci Number

The rule that generates Fibonacci numbers–*a Fibonacci number equals the sum of the two preceding Fibonacci numbers*–is called a **recursive rule** because it defines a number in the sequence using earlier numbers in the sequence. Using subscript notation, the above recursive rule can be expressed by the simple and concise formula

$F_N = F_{N-1} + F_{N-2}$.

Fibonacci Number

There is one thing still missing. The formula $F_N = F_{N-1} + F_{N-2}$ requires two consecutive Fibonacci numbers before it can be used and therefore cannot be applied to generate the first two Fibonacci numbers, F_1 and F_2. For a complete definition we must also explicitly give the values of the first two Fibonacci numbers, namely $F_1 = 1$ and $F_2 = 1$. These first two values serve as "anchors" for the recursive rule and are called the *seeds* of the Fibonacci sequence.

 PEARSON

FIBONACCI NUMBERS (RECURSIVE DEFINITION)

- $F_1 = 1$, $F_2 = 1$ (the *seeds*)
- $F_N = F_{N-1} + F_{N-2}$ (the *recursive rule*)

 PEARSON

Example: Cranking Out Large Fibonacci Numbers

How could one find the value of F_{100}? With a little patience (and a calculator) we could use the recursive definition as a "crank" that we repeatedly turn to ratchet our way up the sequence:

From the seeds F_1 and F_2 we compute F_3, then use F_3 and F_4 to compute F_5, and so on. If all goes well, after many turns of the crank (we will skip the details) you will eventually get to

F_{97} = 83,621,143,489,848,422,977

 PEARSON

Example: Cranking Out Large Fibonacci Numbers

and then to

F_{98} = 135,301,852,344,706,746,049

one more turn of the crank gives

F_{99} = 218,922,995,834,555,169,026

and the last turn gives

F_{100} = 354,224,848,179,261,915,075

converting to dollars yields

$3,542,248,481,792,619,150.75

 PEARSON

Example: Cranking Out Large Fibonacci Numbers

$3,542,248,481,792,619,150.75

How much money is that? If you take $100 billion for yourself and then divide what's left evenly among every man, woman, and child on Earth (about 6.7 billion people), each person would get more than $500 million!

 PEARSON

Leonard Euler

In 1736 Leonhard Euler discovered a formula for the Fibonacci numbers that does not rely on previous Fibonacci numbers.

The formula was lost and rediscovered 100 years later by French mathematician and astronomer Jacques Binet, who somehow ended up getting all the credit, as the formula is now known as Binet's formula.

 PEARSON

BINET'S FORMULA

$$F_N = \left(\frac{1}{\sqrt{5}}\right)\left[\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N\right]$$

Using a Programmable Calculator

You can use the following shortcut of Binet's formula to quickly find the *N*th Fibonacci number for large values of N:

Step 1 Store $A = \left(1+\sqrt{5}\right)/2$ in the calculator's memory.

Step 2 Compute A^N.

Step 3 Divide the result in step 2 by $\sqrt{5}$.

Step 4 Round the result in Step 3 to the nearest whole number. This will give you F_N.

Example: Computing Large Fibonacci Numbers: Part 2

Use the shortcut to Binet's formula with a programmable calculator to compute F_{100}.

Step 1 Compute $\left(1+\sqrt{5}\right)/2$. The calculator should give something like: 1.6180339887498948482.

Step 2 Using the power key, raise the previous number to the power 100. The calculator should show 792,070,839,848,372,253,127.

Example: Computing Large Fibonacci Numbers: Part 2

Step 3 Divide the previous number by $\sqrt{5}$. The calculator should show 354,224,848,179,261,915,075.

Step 4 The last step would be to round the number in Step 3 to the nearest whole number.

In this case the decimal part is so tiny that the calculator will not show it, so the number already shows up as a whole number and we are done.

Why Fibonacci Numbers Are Special

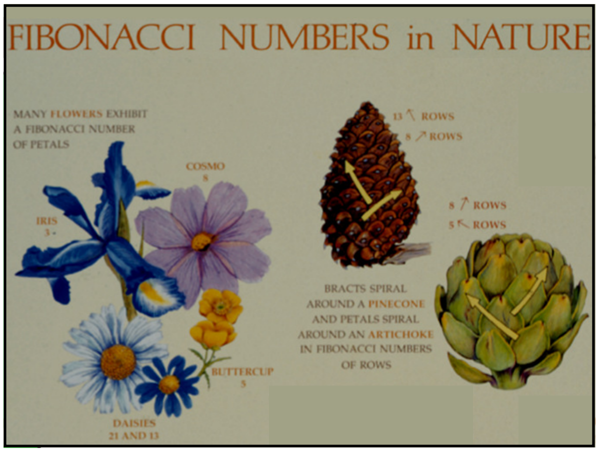
We find Fibonacci numbers when we count the number of petals in certain varieties of flowers: lilies and irises have 3 petals; buttercups and columbines have 5 petals; cosmos and rue anemones have 8 petals; yellow daisies and marigolds have 13 petals; English daisies and asters have 21 petals; oxeye daisies have 34 petals, and there are other daisies with 55 and 89 petals

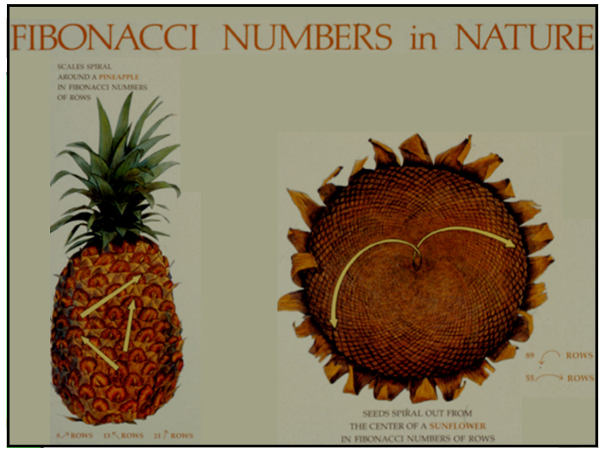
Why Fibonacci Numbers Are Special

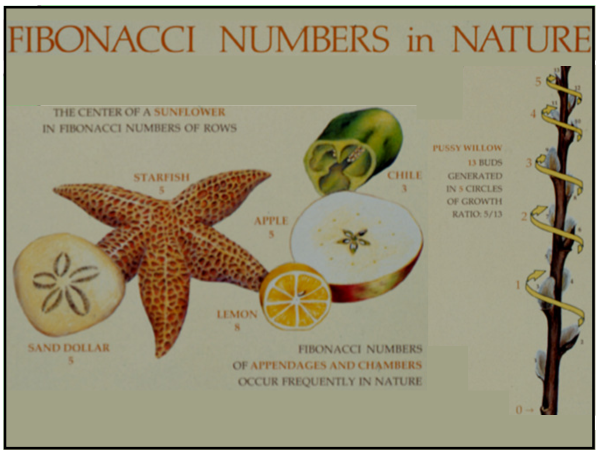
Fibonacci numbers also appear consistently in conifers, seeds, and fruits. The bracts in a pinecone, for example, spiral in two different directions in 8 and 13 rows; the scales in a pineapple spiral in three different directions in 8, 13, and 21 rows; the seeds in the center of a sunflower spiral in 55 and 89 rows.

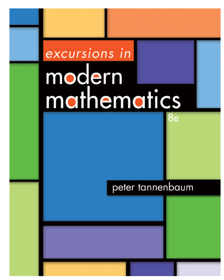
Is it all a coincidence?

Hardly.









Chapter 13: Fibonacci Numbers and the Golden Ratio

13.2 The Golden Ratio

Golden Ratio

This number is one of the most famous and most studied numbers in all mathematics. The ancient Greeks gave it mystical properties and called it the *divine proportion*, and over the years, the number has taken many different names: the *golden number*, the *golden section*, and in modern times the **golden ratio**, the name that we will use from here on. The customary notation is to use the Greek lowercase letter ϕ (phi) to denote the golden ratio.

Golden Ratio

The golden ratio is an irrational number–it cannot be simplified into a fraction, and if you want to write it as a decimal, you can only approximate it to so many decimal places.

For most practical purposes, a good enough approximation is 1.618.

THE GOLDEN RATIO

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$\phi \approx 1.618$$

The sequence of numbers shown above is called the **Fibonacci sequence**, and the individual numbers in the sequence are known as the **Fibonacci numbers**.

The Golden Property

Find a positive number such that when you add 1 to it you get the square of the number.

To solve this problem we let x be the desired number. The problem then translates into solving the quadratic equation $x^2 = x + 1$. To solve this equation we first rewrite it in the form $x^2 - x - 1 = 0$ and then use the quadratic formula. In this case the quadratic formula gives the solutions

$$\frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

The Golden Property

Of the two solutions, one is negative

$\left[(1 \pm \sqrt{5})/2 \approx -0.618\right]$ and the other is the golden ratio $\phi = (1 \pm \sqrt{5})/2.$ It follows that ϕ is *the only positive number with the property that when you add one to the number you get the square of the number*, that is, $\phi^2 = \phi + 1$.

We will call this property the **golden property**. As we will soon see, the golden property has really important algebraic and geometric implications.

Fibonacci Numbers - Golden Property

We will use the golden property $\phi^2 = \phi + 1$ to recursively compute higher and higher powers of ϕ. Here is how:

If we multiply both sides of $\phi^2 = \phi + 1$ by ϕ, we get

$$\phi^3 = \phi^2 + \phi$$

Replacing ϕ^2 by $\phi + 1$ on the RHS gives

$$\phi^3 = (\phi + 1) + \phi = 2\phi + 1$$

Fibonacci Numbers - Golden Property

If we multiply both sides of $\phi^3 = 2\phi + 1$ by ϕ, we get

$$\phi^4 = 2\phi^2 + \phi$$

Replacing ϕ^2 by $\phi + 1$ on the RHS gives

$$\phi^4 = 2(\phi + 1) + \phi = 3\phi + 2$$

Fibonacci Numbers - Golden Property

If we multiply both sides of $\phi^4 = 3\phi + 2$ by ϕ, we get

$$\phi^5 = 3\phi^2 + 2\phi$$

Replacing ϕ^2 by $\phi + 1$ on the RHS gives

$$\phi^5 = 3(\phi + 1) + 2\phi = 5\phi + 3$$

Fibonacci Numbers - Golden Property

If we continue this way, we can express every power of ϕ in terms of ϕ:

$$\phi^6 = 8\phi + 5$$
$$\phi^7 = 13\phi + 8$$
$$\phi^8 = 21\phi + 13 \text{ and so on.}$$

Notice that on the right-hand side we always get an expression involving two consecutive Fibonacci numbers. The general formula that expresses higher powers of in terms of and Fibonacci numbers is as follows.

 PEARSON 13.1-31

POWERS OF THE GOLDEN RATIO

$$\phi^N = F_N\phi + F_{N-1}$$

 PEARSON 13.1-32

Ratio: Consecutive Fibonacci Numbers

We will now explore what is probably the most surprising connection between the Fibonacci numbers and the golden ratio. Take a look at what happens when we take the ratio of consecutive Fibonacci numbers. The table that appears on the following two slides shows the first 16 values of the ratio F_N / F_{N-1}.

 PEARSON 13.1-33

Ratio: Consecutive Fibonacci Numbers

TABLE 9-1 Ratios of Consecutive Fibonacci Numbers

N	F_N	F_{N-1}	F_N/F_{N-1}
2	1	1	1/1 = 1
3	2	1	2/1 = 2
4	3	2	3/2 = 1.5
5	5	3	5/3 = 1.666 . . .
6	8	5	8/5 = 1.6
7	13	8	13/8 = 1.625
8	21	13	21/13 = 1.61538 . . .
9	34	21	34/21 = 1.61904 . . .

 PEARSON 13.1-34

Ratio: Consecutive Fibonacci Numbers

N	F_N	F_{N-1}	F_N/F_{N-1}
10	55	34	55/34 = 1.61764 . . .
11	89	55	89/55 = 1.61818
12	144	89	144/89 = 1.61797 . . .
13	233	144	233/144 = 1.61805 . . .
14	377	233	377/233 = 1.61802 . . .
15	610	377	610/377 = 1.61803 . . .
16	987	610	987/610 = 1.61803 . . .
17	1597	987	1597/987 = 1.61803 . . .

 PEARSON 13.1-35

Ratio: Consecutive Fibonacci Numbers

The table shows an interesting pattern:

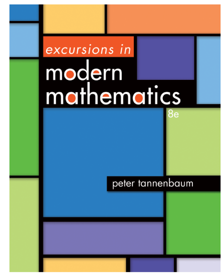
As N gets bigger, the ratio of consecutive Fibonacci numbers appears to settle down to a fixed value, and that *fixed value turns out to be the golden ratio*!

 PEARSON 13.1-36

RATIO OF CONSECUTIVE FIBONACCI NUMBERS

$F_N / F_{N-1} \approx \phi$

and the larger the value of N, the better the approximation.



Chapter 13: Fibonacci Numbers and the Golden Ratio

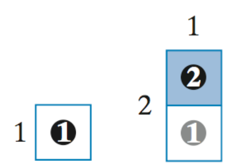
13.4 Spiral Growth in Nature

Spiral Growth in Nature

In nature, where form usually follows function, the perfect balance of a golden rectangle shows up in spiral-growing organisms, often in the form of consecutive Fibonacci numbers. To see how this connection works, consider the following example, which serves as a model for certain natural growth processes.

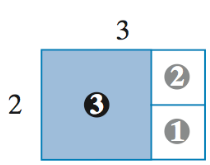
Example: Stacking Squares on Fibonacci Rectangles

Start with a 1 by 1 square. Attach to it a 1 by 1 square. Squares 1 and 2 together form a 1 by 2 Fibonacci rectangle. We will call this the "second generation" shape.



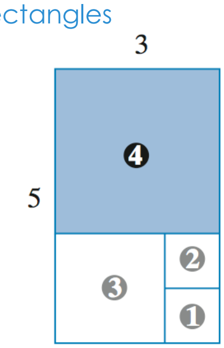
Example: Stacking Squares on Fibonacci Rectangles

For the third generation, tack on a 2 by 2 square (3). The "third-generation" shape is the 3 by 2 Fibonacci rectangle.



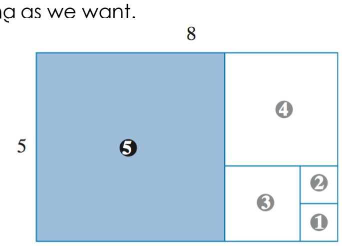
Example: Stacking Squares on Fibonacci Rectangles

Next, tack on a 3 by 3 square, giving a 3 by 5 Fibonacci rectangle.



Example: Stacking Squares on Fibonacci Rectangles

Tacking on a 5 by 5 square results in an 8 by 5 Fibonacci rectangle. You get the picture–we can keep doing this as long as we want.



 PEARSON 13.1-43

Example: Stacking Squares on Fibonacci Rectangles

We might imagine these growing Fibonacci rectangles as a living organism. At each step, the organism grows by adding a square (a very simple, basic shape).

The interesting feature of this growth is that as the Fibonacci rectangles grow larger, they become very close to golden rectangles, and become essentially similar to one another.

This kind of growth–getting bigger while maintaining the same overall shape–is characteristic of the way many natural organisms grow.

 PEARSON 13.1-44

Example: Growth of a "Chambered" Fibonacci Rectangle

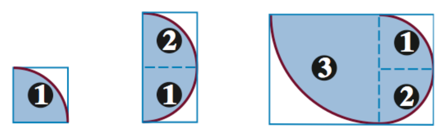
Let's revisit the growth process of the previous example, except now let's create within each of the squares being added an interior "chamber" in the form of a quarter-circle.

We need to be a little more careful about how we attach the chambered square in each successive generation, but other than that, we can repeat the sequence of steps in the previous example to get the sequence of shapes shown on the next two slides.

 PEARSON 13.1-45

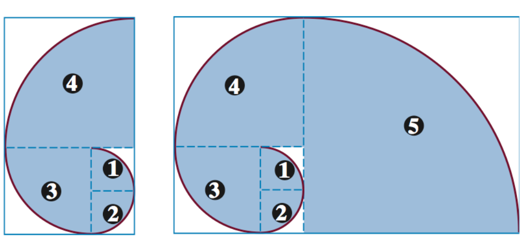
Example: Growth of a "Chambered" Fibonacci Rectangle

These figures depict the consecutive generations in the evolution of the chambered Fibonacci rectangle.



 PEARSON 13.1-46

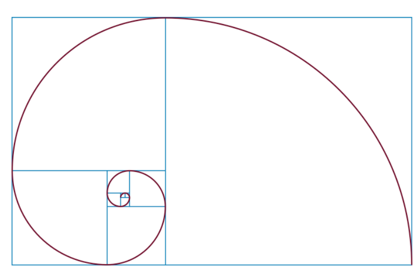
Example: Growth of a "Chambered" Fibonacci Rectangle



 PEARSON 13.1-47

Example: Growth of a "Chambered" Fibonacci Rectangle

The outer spiral formed by the circular arcs is often called a Fibonacci spiral.



 PEARSON 13.1-48

Gnomonic Growth

Natural organisms grow in essentially two different ways.

Humans, most animals, and many plants grow following what can informally be described as an all-around growth rule.

In this type of growth, all living parts of the organism grow simultaneously–but not necessarily at the same rate.

Gnomonic Growth

One characteristic of this type of growth is that there is no obvious way to distinguish between the newer and the older parts of the organism. In fact, the distinction between new and old parts does not make much sense.

Contrast this with the kind of growth exemplified by the shell of the chambered nautilus, a ram's horn, or the trunk of a redwood tree.

Gnomonic Growth

These organisms grow following a one-sided or asymmetric growth rule, meaning that the organism has a part added to it (either by its own or outside forces) in such a way that the old organism together with the added part form the new organism.

At any stage of the growth process, we can see not only the present form of the organism but also the organism's entire past.

Gnomonic Growth

All the previous stages of growth are the building blocks that make up the present structure. The other important aspect of natural growth is the principle of self-similarity: Organisms like to maintain their overall shape as they grow. This is where gnomons come into the picture. For the organism to retain its shape as it grows, the new growth must be a gnomon of the entire organism. We will call this kind of growth process **gnomonic growth**.

Example: Circular Gnomonic Growth

We know from the previous example that the gnomon to a circular disk is an O-ring with an inner radius equal to the radius of the circle.

We can thus have circular gnomonic growth by the regular addition of O-rings.

Example: Circular Gnomonic Growth

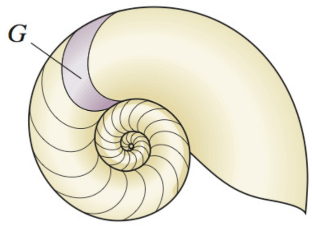
O-rings added one layer at a time to a starting circular structure preserve the circular shape through-out the structure's growth.

When carried to three dimensions, this is a good model for the way the trunk of a redwood tree grows.

And this is why we can "read" the history of a felled redwood tree by studying its rings.

Example: Spiral Gnomonic Growth

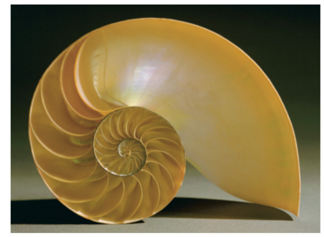
The figure shows a diagram of a cross section of the chambered nautilus. The chambered nautilus builds its shell in stages, each time adding another chamber to the already existing shell.



 PEARSON 13.1-55

Example: Spiral Gnomonic Growth

At every stage of its growth, the shape of the chambered nautilus shell remains the same–the beautiful and distinctive spiral.



 PEARSON 13.1-56

Example: Spiral Gnomonic Growth

This is a classic example of gnomonic growth–each new chamber added to the shell is a gnomon of the entire shell.

The gnomonic growth of the shell proceeds, in essence, as follows:

Starting with its initial shell (a tiny spiral similar in all respects to the adult spiral shape), the animal builds a chamber (by producing a special secretion around its body that calcifies and hardens).

The resulting, slightly enlarged spiral shell is similar to the original one.

 PEARSON 13.1-57

Example: Spiral Gnomonic Growth

The process then repeats itself over many stages, each one a season in the growth of the animal.

Each new chamber adds a gnomon to the shell, so the shell grows and yet remains similar to itself.

This process is a real-life variation of the mathematical spiral-building process discussed previously.

The curve generated by the outer edge of a nautilus shell–a cross section is called a logarithmic spiral.

 PEARSON 13.1-58

Complex Gnomonic Growth

More complex examples of gnomonic growth occur in sunflowers, daisies, pineapples, pinecones, and so on.

Here, the rules that govern growth are somewhat more involved, but Fibonacci numbers and the golden ratio once again play a prominent role.

 PEARSON 13.1-59