Chapter 12: Fractal Geometry

12.1 The Koch Snowflake and Self-Similarity

Geometric Fractal

Our first example of a geometric fractal is a shape known as the Koch snowflake, named after the Swedish mathematician Helge von Koch (1870–1954).

Like other geometric fractals, the Koch snowflake is constructed by means of a recursive process, a process in which the same procedure is applied repeatedly in an infinite feedback loop—the output at one step becomes the input at the next step.

THE KOCH SNOWFLAKE

Start.
Start with a shaded equilateral triangle. We will refer to this starting triangle as the seed of the Koch snowflake. The size of the seed triangle is irrelevant, so for simplicity we will assume that the sides are of length 1.

Step 1.
To the middle third of each of the sides of the seed add an equilateral triangle with sides of length 1/3. The result is the 12-sided “snowflake”.

Step 2.
To the middle third of each of the 12 sides of the “snowflake” in Step 1, add an equilateral triangle with sides of length one-third the length of that side. The result is a “snowflake” with 12 x 4 = 48 sides, each of length $(1/3)^2 = 1/9$.

For ease of reference, we will call the procedure of adding an equilateral triangle to the middle third of each side of the figure procedure KS.

This will make the rest of the construction a lot easier to describe.

Notice that procedure KS makes each side of the figure “crinkle” into four new sides and that each new side has length one-third the previous side.
Step 3. Apply the procedure KS to the “snowflake” in Step 2. This gives the more elaborate “snowflake” with \(48 \times 4 = 192\) sides, each of length \((1/3)^3 = 1/27\).

Step 4. Apply the procedure KS to the “snowflake” in Step 3. This gives the following “snowflake.”

Steps 5, 6, etc. At each step apply the procedure KS to the “snowflake” obtained in the previous Step.

At every step of this recursive process procedure KS produces a new “snowflake,” but after a while it’s hard to tell that there are any changes. Soon enough, the images become visually stable: To the naked eye there is no difference between one snowflake and the next. For all practical purposes we are seeing the ultimate destination of this trip: the Koch snowflake itself.

One advantage of recursive processes is that they allow for very simple and efficient definitions, even when the objects being defined are quite complicated.

The Koch snowflake, for example, is a fairly complicated shape, but we can define it in two lines using a form of shorthand we will call a replacement rule—a rule that specifies how to substitute one piece for another.

Start
Start with a shaded equilateral triangle. (This is the seed.)

Replacement rule:
Replace each boundary line segment with a “crinkled” version.

Koch Snowflake

(REPLACEMENT RULE)
If we only consider the boundary of the Koch snowflake and forget about the interior, we get an infinitely jagged curve known as the **Koch curve**, or sometimes the **snowflake curve**.

We’ll just randomly pick a small part of this section and magnify it. The surprise (or not!) is that we see nothing new—the small detail looks just like the rough detail.

Magnifying further is not going to be much help. The figure shows a detail of the Koch curve after magnifying it by a factor of almost 100.

This seemingly remarkable characteristic of the Koch curve of looking the same at different scales is called **self-similarity**.

As the name suggests, self-similarity implies that the object is similar to a part of itself, which in turn is similar to an even smaller part, and so on.

In the case of the Koch curve the self-similarity has three important properties:
1. It is infinite (the self-similarity takes place over infinitely many levels of magnification)
2. It is universal (the same self-similarity occurs along every part of the Koch curve)
3. It is exact (we see the exact same image at every level of magnification).

There are many objects in nature that exhibit some form of self-similarity, even if in nature self-similarity can never be infinite or exact.

A good example of natural self-similarity can be found in, of all places, a head of cauliflower. The florets look very much like the head of cauliflower but are not exact clones of it. In these cases we describe the self-similarity as approximate (as opposed to exact) self-similarity.

In fact, one could informally say that the Koch snowflake is a two-dimensional mathematical blueprint for the structure of cauliflower.

One of the most surprising facts about the Koch snowflake is that it has a relatively small area but an infinite perimeter and an infinitely long boundary—a notion that seems to defy common sense.
At each step we replace a side by four sides that are 1/3 as long. Thus, at any given step the perimeter $P$ is 4/3 times the perimeter of the preceding step. This implies that the perimeters keep growing with each step, and growing very fast indeed.

After infinitely many steps the perimeter is infinite.

The area of the Koch snowflake is less than the area of the circle that circumscribes the seed triangle and thus, relatively small.

Indeed, we can be much more specific: The area of the Koch snowflake is exactly 8/5 (or 1.6) times the area of the seed triangle.

Having a very large boundary packed within a small area (or volume) is an important characteristic of many self-similar shapes in nature (long boundaries improve functionality, whereas small volumes keep energy costs down).

The vascular system of veins and arteries in the human body is a perfect example of the tradeoff that nature makes between length and volume: Whereas veins, arteries, and capillaries take up only a small fraction of the body’s volume, their reach, by necessity, is enormous.

Laid end to end, the veins, arteries, and capillaries of a single human being would extend more than 40,000 miles.

- It has exact and universal self-similarity.
- It has an infinite perimeter.
- Its area is 1.6 times the area of the seed triangle.

Chapter 12: Fractal Geometry

12.2 The Sierpinski Gasket and the Chaos Game
With the insight gained by our study of the Koch snowflake, we will now look at another well-known geometric fractal called the **Sierpinski gasket**, named after the Polish mathematician Waclaw Sierpinski (1882–1969).

Just like with the Koch snowflake, the construction of the Sierpinski gasket starts with a solid triangle, but this time, instead of repeatedly adding smaller and smaller versions of the original triangle, we will remove them according to the following procedure:

**THE SIERPINSKI GASKET**

**Step 1.**
Remove the triangle connecting the midpoints of the sides of the seed triangle. This gives the shape shown consisting of three shaded triangles, each a half-scale version of the seed and a hole where the middle triangle used to be. We will call this procedure SG.

**Step 2.**
To each of the three shaded triangles apply procedure SG (“removing the middle” of a triangle). The result is the “gasket” consisting of $3^2 = 9$ triangles, each at one-fourth the scale of the seed triangle, plus three small holes of the same size and one larger hole in the middle.

**Step 3.**
To each of the nine shaded triangles apply procedure SG. The result is the “gasket” consisting of $3^3 = 27$ triangles, each at one-eighth the scale of the original triangle, nine small holes of the same size, three medium-sized holes, and one large hole in the middle.

**Steps 4, 5, etc.**
Apply procedure SG to each shaded triangle in the “gasket” obtained in the previous step.
After a few more steps the figure becomes visually stable—the naked eye cannot tell the difference between the gasket obtained at one step and the gasket obtained at the next step.

At this point we have a good rendering of the Sierpinski gasket itself. In your mind’s eye you can think of the figure on the next slide as a picture of the Sierpinski gasket (in reality, it is the gasket obtained at Step 7 of the construction process).

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We might think that the Sierpinski gasket is made of a huge number of tiny triangles, but this is just an optical illusion—there are no solid triangles in the Sierpinski gasket, just specks of the original triangle surrounded by a sea of white triangular holes.

If we were to magnify any one of those small specks, we would see more of the same—specks surrounded by white triangles.

As a geometric object existing in the plane, the Sierpinski gasket should have an area, but it turns out that its area is infinitesimally small, smaller than any positive quantity. Paradoxical as it may first seem, the mathematical formulation of this fact is that the Sierpinski gasket has zero area.

At the same time, the boundary of the “gaskets” obtained at each step of the construction grows without bound, which implies that the Sierpinski gasket has an infinitely long boundary.

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**Start**

Start with a shaded seed triangle.

**Replacement rule:**

Whenever you see a \( \triangle \), replace it with a \( \triangle \).

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As with the Koch curve, the self-similarity of the Sierpinski gasket is (1) infinite, (2) universal, and (3) exact.
PROPERTIES OF
THE SIERPINSKI
GASKET
- It has exact and universal self-similarity.
- It has infinitely long boundary.
- It has zero area.

APPLICATION OF SIERPINSKI GASKET
There is a surprising cutting-edge application of the Sierpinski gasket to one of the most vexing problems of modern life: the inconsistent reception on your cell phone.

A really efficient cell phone antenna must have a very large boundary (more boundary means stronger reception), a very small area (less area means more energy efficiency), and exact self-similarity (self-similarity means that the antenna works equally well at every frequency of the radio spectrum).

APPLICATION OF SIERPINSKI GASKET
What better choice than a design based on a Sierpinski gasket?
Fractal antennas based on this concept are now used not only in cell phones but also in wireless modems and GPS receivers.

THE CHAOS GAME
Now we will introduce a recursive construction rule that involves the element of chance.
The construction is known as the chaos game and is attributed to Michael Barnsley, a mathematician at the Australian National University.

THE CHAOS GAME
We start with an arbitrary triangle with vertices A, B, and C and an honest die. Before we start we assign two of the six possible outcomes of rolling the die to each of the vertices of the triangle. For example, if we roll a 1 or a 2, then A is the chosen vertex; if we roll a 3 or a 4, then B is the chosen vertex; and if we roll a 5 or a 6, then C is the chosen vertex. We are now ready to play the game.

Start.
Roll the die and mark the chosen vertex. Say we roll a 5. This puts us at vertex C.

C
A
B
THE CHAOS GAME

Step 1.
Roll the die again. Say we roll a 2, so the new chosen vertex is A. We now move to the point $M_1$, halfway between the previous position C and the winning vertex A. Mark the new position $M_1$.

THE CHAOS GAME

Step 2.
Roll the die again, and move to the point halfway between the last position $M_1$ and the new chosen vertex A. Say we roll a 3—the move then is to $M_2$, halfway between $M_1$ and B. Mark the new position $M_2$.

Steps 3, 4, etc.
Continue rolling the die, each time moving to a point halfway between the last position and the chosen vertex and marking that point.

Chaos Game after 5000 rolls
What happens after you roll the die many times?
The figure shows the pattern of points after 50 rolls of the die—just a bunch of scattered dots—then after 500, and 5000 rolls.

Chaos Game into Sierpinski Gasket
The last figure is unmistakable: a Sierpinski gasket!
The longer we play the chaos game, the closer we get to a Sierpinski gasket.
After 100,000 rolls of the die, it would be impossible to tell the difference between the two.

Sierpinski Gasket
After a few more steps the figure becomes visually stable—the naked eye cannot tell the difference between the gasket obtained at one step and the gasket obtained at the next step.
At this point we have a good rendering of the Sierpinski gasket itself.
This is a truly surprising turn of events.

The chaos game is ruled by the laws of chance; thus, we would expect that essentially a random pattern of points would be generated.

Instead, we get an approximation to the Sierpinski gasket, and the longer we play the chaos game, the better the approximation gets.

An important implication of this is that it is possible to generate geometric fractals using simple rules based on the laws of chance.

Clearly, the best way to see the chaos game in action is with a computer.

There are many good Web sites for the chaos game and its variations, and a partial list of them is given in the references at the end of the chapter in the textbook.

Our next construction is a variation of the original Sierpinski gasket. For lack of a better name, we will call it the twisted Sierpinski gasket. The construction starts out exactly like the one for the regular Sierpinski gasket, with a seed triangle.

We cut out the middle triangle, whose vertices we will call $M$, $N$, and $L$. The next move (which we will call the "twist") is new. Each of the points $M$, $N$, and $L$ is moved a small amount in a random direction—as if jolted by an earthquake—to new positions $M'$, $N'$, and $L'$.
Procedure TSG

For convenience, we will use the term procedure TSG to describe the combination of the two moves (“cut” and then “twist”).

- **Cut.** Remove the “middle” of the triangle.

- **Twist.** Translate each of the midpoints of the sides by a small random amount and in a random direction.

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**THE TWISTED SIERPINSKI GASKET**

Start.

Start with a shaded seed triangle.

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**THE TWISTED SIERPINSKI GASKET**

**Step 1.**

Apply procedure TSG to the seed triangle. This gives the “twisted gasket” with three twisted triangles and a (twisted) hole in the middle.

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**THE TWISTED SIERPINSKI GASKET**

**Step 2.**

To each of the three shaded triangles in apply procedure TSG. The result is the “twisted gasket” consisting of nine twisted triangles and four holes of various sizes.

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**THE TWISTED SIERPINSKI GASKET**

Steps 3, 4, etc.

Apply procedure TSG to each shaded triangle in the “twisted gasket” obtained in the previous step.

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Twisted Sierpinski Gasket - Step 7

This figure shows Step 7 of the construction of a twisted Sierpinski gasket. Even in this basic form it is remarkable how much the figure resembles a snow-covered mountain.
Real versus Computer Generated

Top is real.
Bottom is CGI.

Add a few of the standard tools of computer graphics—color, lighting, and shading—and we can get a very realistic-looking mountain.

Twisted Sierpinski Gasket

The twisted Sierpinski gasket has infinite and universal self-similarity, but due to the randomness of the twisting process the self-similarity is only approximate—when we magnify any part of the gasket we see similar but not identical images.

Twisted Sierpinski Gasket

It is the approximate self-similarity that gives this shape that natural look that the original Sierpinski gasket lacks, and it illustrates why randomness is an important element in creating artificial imitations of natural-looking shapes.

The Mandelbrot Set

In this section we will introduce one of the most interesting and beautiful geometric fractals ever created, an object called the Mandelbrot set after Benoît Mandelbrot, a mathematician at Yale University.

Some of the mathematics behind the Mandelbrot set goes a bit beyond the level of this book, so we will describe the overall idea and try not to get bogged down in the details.
We will start this section with a brief visual tour. The Mandelbrot set is the shape shown, a strange-looking blob of black. In the wild imagination of some, the Mandelbrot set looks like some sort of bug—an exotic extraterrestrial flea. The “flea” is made up of a heart-shaped body (called a cardioid), a head, and an antenna coming out of the middle of the head. A careful look shows that the flea has many “smaller fleas that prey on it.”

We can only begin to understand the full extent of the infestation when we look at the finely detailed close-up of the boundary of the Mandelbrot set. The Mandelbrot set has some strange form of infinite and approximate self-similarity—at infinitely many levels of magnification we see the same theme—similar but never identical fleas surrounded by similar smaller fleas.

When we magnify the view around the boundary even further, we can see that these secondary fleas have fleas that “prey on them.” At the same time, we can see that there is tremendous variation in the regions surrounding the individual fleas. The images we see are a peek into a psychedelic coral reef—a world of strange “urchins” and “seahorse tails.”

When we magnify the view around the boundary even further, we view a virtual sea of “anemone” and “starfish.” Further magnification shows an even more exotic and beautiful landscape of one of the seahorse tails.
The Mandelbrot Set

A further close-up of a section from the previous slide.

An even further magnification of it is seen in revealing a tiny copy of the Mandelbrot set surrounded by a beautiful arrangement of swirls, spirals, and seahorse tails.

The Mandelbrot Set

Anywhere we choose to look at these pictures, we will find (if we magnify enough) copies of the original Mandelbrot set, always surrounded by an infinitely changing, but always stunning, background.

The infinite, approximate self-similarity of the Mandelbrot set manages to blend infinite repetition and infinite variety, creating a landscape as consistently exotic and diverse as nature itself.

How does this magnificent mix of beauty and complexity called the Mandelbrot set come about? Incredibly, the Mandelbrot set itself can be described mathematically by a recursive process involving simple computations with complex numbers.

The basic building block for complex numbers is the number \( i = \sqrt{-1} \).

Starting with \( i \) we can build all other complex numbers: \( 3 + 2i, -0.125 + 0.75i, 1 + 0i = 1 \).

For our purposes, the most important fact about complex numbers is that they have a geometric interpretation: The complex number \((a + bi)\) can be identified with the point \((a, b)\) in a Cartesian coordinate system.

This identification means that every complex number can be thought of as a point in the plane and that operations with complex numbers have geometric interpretations.

The key concept in the construction of the Mandelbrot set is that of a Mandelbrot sequence.

**A Mandelbrot sequence** is an infinite sequence of complex numbers that starts with an arbitrary complex number \( s \) we call the seed, and then each successive term in the sequence is obtained recursively by adding the seed \( s \) to the square of the previous term.

**Seed**

Step 1

\[ \cdots \]

Step 2

\[ \cdots \]

Step 3

\[ \cdots \]
**Start.**
Choose an arbitrary complex number $s$. We will call $s$ the seed of the Mandelbrot sequence. Set the seed $s$ to be the initial term of the sequence ($s_0 = s$).

**Recursive Rule:**
To find the next term in the sequence, square the preceding term and add the seed [$s_{n+1} = (s_n)^2 + s$].

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**Example: Escaping Mandelbrot Sequences**
The figure shows the first few terms of the Mandelbrot sequence with seed $s = 1$. Since integers and decimals are also complex numbers, they make perfectly acceptable seeds.

- **Seed**: $s = 1$
- **Step 1**: $s_1 = s^2 + 1 = 2$
- **Step 2**: $s_2 = 2^2 + 1 = 5$
- **Step 3**: $s_3 = 5^2 + 1 = 26$
- **Step 4**: $s_4 = 26^2 + 1 = 677$

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**Example: Escaping Mandelbrot Sequences**
In general, when the points that represent the terms of a Mandelbrot sequence move farther and farther away from the origin, we will say that the Mandelbrot sequence is escaping.

The basic rule that defines the Mandelbrot set is that seeds of escaping Mandelbrot sequences are not in the Mandelbrot set and must be assigned some color other than black.

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**Example: Escaping Mandelbrot Sequences**
While there is no specific rule that tells us what color should be assigned, the overall color palette is based on how fast the sequence is escaping.

The typical approach is to use "hot" colors such as reds, yellows, and oranges for seeds that escape slowly and "cool" colors such as blues and purples for seeds that escape quickly.
**Example: Escaping Mandelbrot Sequences**

The seed for $s = 1$ quickly escapes very quickly, and the corresponding point in the Cartesian plane is painted blue. The seed is assigned a "cool" color (blue).

**Example: Periodic Mandelbrot Sequences**

The figure shows the first few terms of the Mandelbrot sequence with seed $s = -1$. The pattern that emerges here is also clear—the numbers in the sequence alternate between 0 and -1. In this case, we say that the Mandelbrot sequence is periodic.

### Seed Steps

1. $s_1 = (-1)^2 + (-1) = 0$
2. $s_2 = 0^2 + (-1) = -1$
3. $s_3 = (-1)^2 + (-1) = 0$

**Example: Periodic Mandelbrot Sequences**

In general, a Mandelbrot sequence is said to be periodic if at some point the numbers in the sequence start repeating themselves in a cycle. When the Mandelbrot sequence is periodic, the seed is a point of the Mandelbrot set and thus is assigned the color black.

**Example: Attached Mandelbrot Sequences**

Here the growth pattern is not obvious, and additional terms of the sequence are needed.

Further computation (a calculator will definitely come in handy) shows that as we go farther and farther out in this sequence, the terms get closer and closer to the value $-0.5$. In this case, we will say that the sequence is attracted to the value $-0.5$.

**Example: Attached Mandelbrot Sequences**

In general, when the terms in a Mandelbrot sequence get closer and closer to a fixed complex number $a$, we say that $a$ is an attractor for the sequence or, equivalently, that the sequence is attracted to $a$.

Just as with periodic sequences, when a Mandelbrot sequence is attracted, the seed is in the Mandelbrot set and is colored black.
Example: A Periodic Mandelbrot Sequence with Complex Terms

In this example we will examine the growth of the Mandelbrot sequence with seed $s = i$.

Seed

$\begin{align*}
    s &= i \\
    s_1 &= i^2 + i \\
    s_2 &= (1 + i)^2 + i \\
    s_3 &= (-1 + i)^2 + i \\
    s_4 &= -1 + i \\
    s_5 &= -1 + i \\
    s_6 &= -1 + i \\
    \ldots
\end{align*}$

Example: A Periodic Mandelbrot Sequence with Complex Terms

At this point we notice that $s_3 = s_1$, which implies $s_4 = s_2$, $s_5 = s_3$, and so on. This, of course, means that this Mandelbrot sequence is periodic, with its terms alternating between the complex numbers $-1 + i$ (odd terms) and $-i$ (even terms).

The key conclusion from the preceding computations is that the seed $i$ is a black point inside the Mandelbrot set.

Example: A Mandelbrot Sequence with Three Complex Attractors

We will examine the growth of the Mandelbrot sequence with seed $s = -0.125 + 0.75i$.

$s_0 = -0.125 + 0.75i$
$s_1 = -0.671875 + 0.5625i$
$s_2 = 0.0100098 - 0.00585938i$
$s_3 = -0.124934 + 0.749883i$
$s_4 = -0.671716 + 0.562628i$
$s_5 = 0.00965136 - 0.00585206i$
$s_6 = -0.124941 + 0.749887i$

We can see that the terms in this Mandelbrot sequence are complex numbers that essentially cycle around in sets of three and are approaching three different attractors.

Since this Mandelbrot sequence is attracted, the seed $s = -0.125 + 0.75i$ represents another point in the Mandelbrot set.

Example: A Mandelbrot Sequence with Three Complex Attractors

Definition: The Mandelbrot Set

If the Mandelbrot sequence is periodic or attracted, the seed is a point of the Mandelbrot set and assigned the color black; if the Mandelbrot sequence is escaping, the seed is a point outside the Mandelbrot set and assigned a color that depends on the speed at which the sequence is escaping (hot colors for slowly escaping sequences, cool colors for quickly escaping sequences).

Definition: The Mandelbrot Set

There are a few technical details that we omitted, but essentially these are the key ideas behind the amazing pictures that we saw in the beginning of this section.

Because the Mandelbrot set provides a bounty of aesthetic returns for a relatively small mathematical investment, it has become one of the most popular mathematical playthings of our time.