Math 240 – Rimmer

Midterm # 3 – Fall 2007

December 4, 2007

1. Evaluate \[ \iint_S (6x - 3y) \, dA \]

where \( S \) is the region bounded by the lines
\[ 2x - y = 1 \quad 2x - y = 3 \]
\[ x + y = 1 \quad x + y = 2 \]

A) 2    B) 3    C) 12    D) 4    E) 6

\[ u = 2x - y \quad 1 \leq u \leq 3 \]
\[ v = x + y \quad 1 \leq v \leq 2 \]

\[ v = \frac{1}{3}(u + v) + y \Rightarrow y = \frac{2}{3}v - \frac{1}{3}u \Rightarrow -3y = u - 2v \Rightarrow 6x - 3y = 3u \]

\[ u = 2x - y \]
\[ v = x + y \]

\[ J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = \frac{2}{3} + \frac{1}{3} = \frac{1}{3} \]

\[ \iint_S (6x - 3y) \, dA = \int \int 3u |J(u, v)| \, dA = \int_1^3 \int_1^3 3u \, du \, dv = 4 \]
2.
Solve the homogeneous differential equation
\[
\left(1 + \frac{y}{x}\right)dx - dy = 0 \text{ with } y(1) = 0
\]
Find \( f(e) \).

A) \( e \)  B) \( 1 \)  C) \( \frac{1}{e} \)  D) \( \frac{1}{e^2} \)  E) none of the above

\[
y = u x \]
\[
dy = u dx + x du
\]
\[
\frac{y}{x} = u
\]

\[
\left(1 + \frac{y}{x}\right)dx - dy = 0 \Rightarrow (1 + u)dx - (udx + xdu) = 0
\]
\[
(1 + u - u)dx - xdu = 0
\]
\[
dx = xdu
\]
\[
\frac{1}{x} dx = du
\]
\[
u = \ln x + C
\]
\[
\frac{y}{x} = \ln x + C
\]
\[
y = x \ln x + Cx
\]
\[
y(1) = 0 \Rightarrow 0 = \ln 1 + C \Rightarrow C = 0
\]
\[
y = x \ln x
\]
\[
y(e) = e \ln e = e\]
3. Solve the differential equation

\[ y' + y = y^3 \]

with \( y(0) = \frac{1}{3} \).

A) \( \frac{1}{\sqrt{5e^{-2x} + 4}} \)  
B) \( \frac{1}{\sqrt{3e^{-2x} + 6}} \)  
C) \( \frac{1}{\sqrt{8e^{-2x} + 1}} \)  
D) \( \frac{1}{\sqrt{7e^{-2x} + 2}} \)  
E) none of the above

Bernoulli equation with \( n = 3 \)

\[ u = y^{-2} \]
\[ y = u^{-1/2} \text{ and } y^3 = u^{-3/2} \]

\[ \frac{dy}{dx} = -\frac{1}{2} u^{-3/2} \frac{du}{dx} \]
\[ y' + y = y^3 \Rightarrow -\frac{1}{2} u^{-3/2} \frac{du}{dx} + u^{-1/2} = u^{-3/2} \]

Multiply by \( -2u^{3/2} \)
\[ u' - 2u = -2 \iff \text{Linear} \]

integrating factor \( \mu = e^{\int p(x)dx} = e^{-2x} \)

Mult. both sides by \( \mu \):
\[ e^{-2x}u' - 2ue^{-2x} = -2e^{-2x} \]

\[ \left(e^{-2x}u\right)' = -2e^{-2x} \]

Integrate both sides
\[ e^{-2x}u = -2\int e^{-2x} \, dx \]
\[ e^{-2x}u = e^{-2x} + C \]

Solve for \( u \)
\[ u = 1 + Ce^{2x} \]

\[ y^{-2} = 1 + Ce^{2x} \]
\[ y = \left(1 + Ce^{2x}\right)^{-1/2} \]

\[ y = \frac{1}{\sqrt{1 + Ce^{2x}}} \quad y(0) = 1/3 \Rightarrow \frac{1}{3} = \frac{1}{\sqrt{1 + C}} \Rightarrow C = 8 \]

\[ y = \frac{1}{\sqrt{1 + 8e^{2x}}} \]
4. Solve
\[ x^2 y'' - xy' + 2y = 0 \quad y(1) = -1, y'(1) = -1 \]

Find \( y(e^\pi) \).

A) \(-e^\pi\)  B) \(e^\pi\)  C) 0  D) \(-\pi\)  E) \(\pi\)

Cauchy-Euler Equation \( a = 1, b = -1, c = 2 \)

Auxilliary Equation \( \Rightarrow x^r \left( ar^2 + (b - a)r + c \right) = 0 \)
\[ r^2 - 2r + 2 = 0 \]
\[ r = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i \]

\( \cdot \) complex roots \( r_1 = \alpha + \beta i, r_2 = \alpha - \beta i \)
\[ y = x^\alpha \left( c_1 \cos (\beta \ln x) + c_2 \sin (\beta \ln x) \right) \]
\[ y = x( c_1 \cos (\ln x) + c_2 \sin (\ln x) ) \]
\[ y' = ( c_1 \cos (\ln x) + c_2 \sin (\ln x) ) + x \left( -c_1 \sin (\ln x) \cdot \frac{1}{x} + c_2 \cos (\ln x) \cdot \frac{1}{x} \right) \]
\[ y' = \left( (c_1 + c_2) \cos (\ln x) + (c_2 - c_1) \sin (\ln x) \right) \]
\[ y(1) = -1 \Rightarrow -1 = 1 \begin{pmatrix} c_1 \cos (\ln 1) + c_2 \sin (\ln 1) \\ \end{pmatrix} \Rightarrow c_1 = -1 \]
\[ y'(1) = -1 \Rightarrow c_1 + c_2 = -1 \text{ but } c_1 = -1 \Rightarrow c_2 = 0 \]
\[ y = -x \cos (\ln x) \]
\[ y(e^\pi) = -e^\pi \cos (\ln e^\pi) = -e^\pi \cos (\pi) = e^\pi \]
5. A spring with a **mass** of 2 kg, attached has damping constant 16. A force of 12.8 N keeps the spring stretched 0.2 m beyond its natural length. If it starts at equilibrium position with a downward velocity of 2.4 m/s, how much time passes until the spring reaches equilibrium for the first time?

A) \( \frac{\pi}{4} \)  
B) \( \frac{\pi}{2} \)  
C) \( \pi \)  
D) \( 2\pi \)  
E) \( \frac{\pi}{8} \)

\( m = 2, \beta = 16 \)
\( F = ks \Rightarrow 12.8 = 0.2k \Rightarrow k = 64 \)

\[ x'' + \frac{\beta}{m} x' + \frac{k}{m} x = 0 \]

\[ x'' + 8x' + 32x = 0 \]

Let \( 2\lambda = \frac{\beta}{m} \Rightarrow \lambda = 4, \quad \text{and} \quad \omega^2 = \frac{k}{m} \Rightarrow \omega^2 = 32 \)

\[ r = -\lambda \pm \sqrt{\lambda^2 - \omega^2} \Rightarrow r = -4 \pm \sqrt{16 - 32} \Rightarrow r = -4 \pm 4i \]

\[ x(t) = e^{-4t} \left( c_1 \cos \left( \sqrt{\omega^2 - \lambda^2} \right) t + c_2 \sin \left( \sqrt{\omega^2 - \lambda^2} \right) t \right) \]

\[ x(t) = e^{-4t} \left( c_1 \cos (4t) + c_2 \sin (4t) \right) \]

starts at equilibrium position \( \Rightarrow x(0) = 0 \Rightarrow c_1 = 0 \)

\[ x(t) = c_2 e^{-4t} \sin (4t) \]

starts with a downward velocity of 2.4 m/s \( \Rightarrow x'(0) = 2.4 \Rightarrow c_1 = 0 \)

\[ x'(t) = -4c_2 e^{-4t} \sin (4t) + 4c_2 e^{-4t} \cos (4t) \]

\[ x'(0) = 4c_2 = 2.4 \Rightarrow 0.6 \]

\[ x(t) = 0.6e^{-4t} \sin (4t) \]

spring reaches equilibrium for the first time \( \Rightarrow x(t) = 0, t > 0 \)

\[ 0.6e^{-4t} \sin (4t) = 0 \Rightarrow \sin (4t) = 0 \Rightarrow 4t = \pi \Rightarrow t = \frac{\pi}{4} \]
6. Let \( f(t) = L^{-1} \left\{ \frac{2s-3}{s^2+2s+10} \right\} \). Find \( f\left( \frac{\pi}{6} \right) \)

A) \( \frac{5}{3} e^{-\%} \)  
B) \( 2e^{-\%} \)  
C) \( \left( \sqrt{3} - \frac{5}{6} \right) e^{-\%} \)  
D) \( \frac{5}{6} e^{-\%} \)  
E) none of the above

\[
A) \frac{5}{3} e^{-\%} \quad B) 2e^{-\%} \quad C) \left( \sqrt{3} - \frac{5}{6} \right) e^{-\%} \quad D) \frac{5}{6} e^{-\%} \quad E) \text{none of the above}
\]

\[
\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2 + 9} = \frac{2s}{(s+1)^2 + 9} - \frac{3}{(s+1)^2 + 9}
\]

\[
= \frac{2s+2-2}{(s+1)^2 + 9} - \frac{3}{(s+1)^2 + 9} = \frac{2(s+1)}{(s+1)^2 + 9} - \frac{5}{(s+1)^2 + 9} = 2 \frac{(s+1)}{(s+1)^2 + 9} - \frac{5}{3} \frac{\frac{3}{\pi}}{(s+1)^2 + 9}
\]

\[
f(t) = L^{-1} \left\{ \frac{2s-3}{s^2+2s+10} \right\} = L^{-1} \left\{ 2 \frac{(s+1)}{(s+1)^2 + 9} - \frac{5}{3} \frac{\frac{3}{\pi}}{(s+1)^2 + 9} \right\}
\]

\[
f(t) = 2L^{-1} \left\{ \frac{(s+1)}{(s+1)^2 + 9} \right\} - \frac{5}{3} L^{-1} \left\{ \frac{3}{(s+1)^2 + 9} \right\}
\]

\[
f(t) = e^{-t} \left( 2 \cos(3t) - \frac{5}{3} \sin(3t) \right)
\]

\[
f\left( \frac{\pi}{6} \right) = e^{-\pi/6} \left( 2 \cos\left( \frac{\pi}{2} \right) - \frac{5}{3} \sin\left( \frac{\pi}{2} \right) \right) = \left[ \frac{5}{3} e^{-\pi/6} \right]
\]
7. Find the Laplace Transform of

\[ f(t) = \frac{1}{2} te^{-2t} \sin t \]

by any means other than the definition.

A) \( \frac{s + 4}{s^2 + 4s + 5} \)  
B) \( \frac{s + 2}{(s^2 + 4s + 5)^2} \)  
C) \( \frac{2s + 4}{(s^2 + 4s + 5)^2} \)  
D) \( \frac{s + 2}{s^2 - 4s - 5} \)  
E) none of the above

\[
L\{f(t)\} = \frac{1}{2} L\{te^{-2t} \sin t\} = -\frac{1}{2} \frac{d}{dt} \left[L\{e^{-2t} \sin t\}\right]
\]

\[
= -\frac{1}{2} \frac{d}{dt} \left[\frac{1}{(s + 2)^2 + 1}\right] = -\frac{1}{2} \frac{d}{dt} \left[\frac{1}{s^2 + 4s + 5}\right] = -\frac{1}{2} \frac{d}{dt} \left[(s^2 + 4s + 5)^{-1}\right]
\]

\[
= -\frac{1}{2} \left[\frac{d}{dt} \left(s^2 + 4s + 5\right)^{-2} \cdot (2s + 4)\right] = \frac{s + 2}{\left(s^2 + 4s + 5\right)^2}
\]
8. Derive the Laplace transform of 

\[ f(t) = t \cosh(at) \]

You should find a theorem or property more useful than the definition. “I read it off my table” will receive 3 points.

A) \( \frac{-2as}{(s^2 - a^2)^2} \)  
B) \( \frac{2as}{(s^2 - a^2)^2} \)  
C) \( \frac{-\left(s^2 + a^2\right)}{(s^2 - a^2)^2} \)  
D) \( \frac{s^2 + a^2}{(s^2 - a^2)^2} \)  
E) none of the above

\[
\mathcal{L}\{f(t)\} = \mathcal{L}\{t \cosh(at)\} = -\frac{d}{dt}\mathcal{L}\{\cosh(at)\} = -\frac{d}{dt}\left[\frac{s}{s^2 - a^2}\right] = -\left[\frac{1 \cdot (s^2 - a^2) - s \cdot 2s}{(s^2 - a^2)^2}\right] = -\left[\frac{-s^2 - a^2}{(s^2 - a^2)^2}\right] = \frac{s^2 + a^2}{(s^2 - a^2)^2} \]
9. Solve \( y'' - 2y' + 2y = e^{-2t} \) \( y(0) = 0, y'(0) = 0 \)

Using Laplace transforms. Find \( y(2\pi) \).

A) \( \frac{1}{5} e^{3\pi} \)
B) \( 4e^{2\pi} \)
C) \( \frac{4}{5} e^{2\pi} \)
D) \( \frac{e^{2\pi} - e^{-4\pi}}{5} \)
E) none of the above

\[
L\{y'' - 2y' + 2y\} = L\{e^{-2t}\} \quad y(0) = 0, y'(0) = 0
\]

\[
L\{y''\} - 2L\{y'\} + 2L\{y\} = \frac{1}{s + 2}
\]

\[
s^2L\{y\} - sL\{0\} - y'(0) - 2[sL\{y\} - y(0)] + 2L\{y\} = \frac{1}{s + 2}
\]

\[
(s^2 - 2s + 2)L\{y\} = \frac{1}{s + 2}
\]

\[
L\{y\} = \frac{1}{(s + 2)(s^2 - 2s + 2)} = \frac{A}{s + 2} + \frac{Bs + C}{s^2 - 2s + 2}
\]

\[
\Rightarrow A(s^2 - 2s + 2) + (Bs + C)(s + 2) = 1
\]

\[
(A + B)s^2 + (-2A + 2B + C)s + (2A + 2C) = 1
\]

\[
A + B = 0 \Rightarrow B = -A
\]

\[
2A + 2C = 1 \Rightarrow 2C = 1 - 2A \Rightarrow C = \frac{1}{2} - A
\]

\[
-2A + 2B + C = 0 \Rightarrow -2A - 2A + \frac{1}{2} - A = 0 \Rightarrow 5A = \frac{1}{2} \Rightarrow A = \frac{1}{10}
\]

\[
\Rightarrow B = -\frac{1}{10} \text{ and } C = \frac{1}{2} - \frac{1}{10} = \frac{2}{5}
\]

\[
L\{y\} = \frac{1}{10} \left( \frac{1}{s + 2} \right) + \frac{\frac{1}{10} s + \frac{2}{5}}{s^2 - 2s + 2} = \frac{1}{10} \left( \frac{1}{s + 2} \right) + \frac{\frac{1}{10} s}{(s - 1)^2 + 1} + \frac{\frac{2}{5}}{(s - 1)^2 + 1}
\]

\[
L\{y\} = \frac{1}{10} \left( \frac{1}{s + 2} \right) + \frac{-\frac{1}{10} s + \frac{10}{10} - \frac{1}{10}}{(s - 1)^2 + 1} + \frac{\frac{2}{5}}{(s - 1)^2 + 1}
\]

\[
L\{y\} = \frac{1}{10} \left( \frac{1}{s + 2} \right) - \frac{\frac{1}{10} s - \frac{1}{10}}{(s - 1)^2 + 1} + \frac{\frac{3}{10} - \frac{1}{10}}{(s - 1)^2 + 1}
\]

\[
\Rightarrow y = \frac{1}{10} e^{-2t} + e^{-t} \left( -\frac{1}{10} \cos t + \frac{3}{10} \sin t \right)
\]

\[
y(\pi) = \frac{1}{10} e^{-4\pi} + e^{-2\pi} \left( -\frac{1}{10} \right) = \frac{1}{10} \left( e^{-4\pi} - e^{-2\pi} \right)
\]
10. \( x'' - x = f(t) \)

with \( x(0) = 0, x'(0) = 0 \)

Find the value of the Laplace transform of \( x \) evaluated at \( s = 2 \).

(no need to find the function \( x(t) \))

A) \( \frac{1}{2}(1-e^{-2}) \)  
B) \( \frac{1}{6}(1-e^{-2}) \)  
C) \( \frac{8}{3}(1-e^{-1/2}) \)  
D) \( \frac{1}{3}(1-e^{-2}) \)  
E) none of the above

\[
L\{x' - x\} = L\{f(t)\}
\]
\[
L\{x\} - sL\{x\} = L\{f(t)\}
\]
\[
(s^2-1)L\{x\} = L\{f(t)\}
\]
\[
L\{x\} = \frac{L\{f(t)\}}{(s^2 - 1)}
\]
\[
L\{x\} = \frac{1-e^{-s}}{s(s^2 - 1)}
\]

\( L\{x\} \) at \( s = 2 \) \( \Rightarrow \frac{1-e^{-2}}{2(4-1)} = \frac{1}{6}(1-e^{-2}) \)