Section 9.9 Independence of Path

\( P(x, y) \, dx + Q(x, y) \, dy \) is called an **exact differential** if there exists a function \( \phi(x, y) \) called a **potential function** such that \( d\phi = P(x, y) \, dx + Q(x, y) \, dy \)

\[ \Rightarrow \frac{\partial \phi}{\partial x} = P(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = Q(x, y) \]

If this is the case, then the mixed partial of \( \phi \) must be equal.

\[ \Rightarrow \phi_{xy} = P_y \quad \text{should equal} \quad \phi_{yx} = Q_x \]

\[ \left\{ \begin{array}{l}
Pdx + Qdy = \int_C \mathbf{F} \cdot d\mathbf{r} \\
\text{When} \quad \mathbf{F} = \langle P(x, y), Q(x, y) \rangle \quad \text{and} \quad \mathbf{r} = \langle x, y \rangle \\
\Rightarrow \mathbf{F} = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle = \nabla \phi
\end{array} \right. \]

For 3 dimensions:

\[ d\phi = P(x, y, z) \, dx + Q(x, y, z) \, dy + R(x, y, z) \, dz \]

\[ \frac{\partial \phi}{\partial z} = R(x, y, z) \Rightarrow R_z = Q_z \quad \text{and} \quad R_z = P_z \]

\[ \mathbf{F} = \langle P, Q, R \rangle = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle = \nabla \phi \]

\[ \text{Since} \quad \mathbf{F} = \nabla \phi, \quad \text{then} \quad \text{curl} \ \mathbf{F} = \text{curl} (\text{grad} \phi) = 0 \]

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A line integral whose value is the same for every curve connected the starting and ending point is called **independent of path**.

2-dimensions

\[ \int_C Pdx + Qdy \quad \text{independent of path} \iff Pdx + Qdy \text{ is an exact differential} \]

\[ \downarrow \]

\[ P_y = Q_x \]

3-dimensions

\[ \int_C Pdx + Qdy + Rdz \quad \text{independent of path} \iff Pdx + Qdy + Rdz \text{ is an exact differential} \]

\[ \downarrow \]

\[ \mathbf{F} = \langle P, Q, R \rangle \]

\[ \text{curl} \ \mathbf{F} = 0 \]

\( \mathbf{F} \) is called **conservative**
Fundamental Theorem of Calculus

If \( f(x) \) is a continuous function on \([a,b]\) such that \( F(x) \) is a function whose derivative is \( f(x) \) on \([a,b]\), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

Fundamental Theorem of Line Integrals

Let \( C \) be a curve with starting point \((x_0, y_0)\) and ending point \((x_1, y_1)\).
Suppose \( P(x, y) \, dx + Q(x, y) \, dy \) is an exact differential (there exists \( \phi \) such that \( d\phi = P(x, y) \, dx + Q(x, y) \, dy \)).

\[
\int_{C} P(x, y) \, dx + Q(x, y) \, dy = \phi(x_1, y_1) - \phi(x_0, y_0)
\]

\[
\int_{C} P\, dx + Q\, dy \text{ independent of path}
\]

\[\text{a) Find } \phi \text{ and evaluate it at the endpoints of } C\]

\[\text{b) Pick a "convenient" path and evaluate the line integral. (convenient = horizontal or vertical)}\]

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**Section 9.8 #14**

Evaluate \( \int_{C} y\, dx + x\, dy \)

on \( C \): line segments from \((0,0)\) to \((1,0)\)
and from \((1,0)\) to \((1,1)\)

\[
\int_{C} y\, dx + x\, dy = \int_{C_1} y\, dx + x\, dy + \int_{C_2} y\, dx + x\, dy
\]

\[
C_1: \quad x = t, \quad y = 0, \quad 0 \leq t \leq 1 \quad C_2: \quad x = 1, \quad y = t, \quad 0 \leq t \leq 1
\]

\[
\int_{C_1} y\, dx + x\, dy = \int_{0}^{1} (0 + 0) \, dt = 0 \quad \int_{C_2} y\, dx + x\, dy = \int_{0}^{1} (0 + 1) \, dt = 1
\]

\[
\int_{C} y\, dx + x\, dy = 1
\]

Evaluate \( \int_{C_1} y\, dx + x\, dy \)

on \( C_1 \):

\[
C_1: \quad x = t, \quad y = t, \quad 0 \leq t \leq 1
\]

\[
\int_{C_1} y\, dx + x\, dy = \int_{0}^{1} t \, dt + \int_{0}^{1} t \, dt = \frac{1}{2} \int_{0}^{1} 2 \, dt = 1
\]

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Section 9.9 #12

Determine whether the given vector field is a gradient field. If so, find a potential function \( \phi \) for \( \mathbf{F} \).

\[
\mathbf{F} = 2xy\mathbf{i} + 3y^2(x^2 + 1)\mathbf{j}
\]

\[
P = 2xy^3 \quad Q = 3y^2(x^2 + 1)
\]

For \( \mathbf{F} \) to be a gradient field, we need \( P_y = Q_x \), so \( \mathbf{F} \) is a gradient field.

\[
\phi(x, y) = \int P
dx = \int 2xy\,dx = x^2y^3 + G(y)
\]

\[
\phi(x, y) = \int Q
dy = \int 3y^2(x^2 + 1)\,dy = y^3(x^2 + 1) + H(x)
\]

\[
\Rightarrow x^2y^3 + G(y) = y^3(x^2 + 1) + H(x)
\]

\[
\Rightarrow H(x) = C_1
\]

\[
\phi(x, y) = x^2y^3 + y^3 + C
\]

Section 9.9 #18

Find the work done by the force \( \mathbf{F}(x, y) = (2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j} \) along the indicated curve.

\[
P = 2x + e^{-y} \quad Q = 4y - xe^{-y}
\]

\[
P_y = -e^{-y} \quad Q_x = -e^{-y} \quad \Rightarrow \text{so } \mathbf{F} \text{ is a gradient field.}
\]

\[
\phi(x, y) = \int P
dx = \int (2x + e^{-y})\,dx = x^2 + xe^{-y} + G(y)
\]

\[
\phi(x, y) = \int Q
dy = \int (4y - xe^{-y})\,dy = 2y^2 + xe^{-y} + H(x)
\]

\[
\Rightarrow x^2 + xe^{-y} + G(y) = 2y^2 + xe^{-y} + H(x)
\]

\[
\Rightarrow H(x) = x^2 + C_1 \quad \text{and } G(y) = 2y^2 + C_2
\]

\[
\phi(x, y) = x^2 + xe^{-y} + 2y^2 + C
\]

\[
\text{Work} = \int_C \mathbf{F} \cdot \mathbf{dr} = \phi(x, y)|_{(2,0)}^{(2,0)}
\]

\[
= \left(x^2 + xe^{-y} + 2y^2\right)|_{(2,0)}^{(2,0)}
\]

\[
= (4 - 2) - (4 + 2) = -4
\]

“Convenient” Path

\[
\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C (2x + e^{-y})\,dx + (4y - xe^{-y})\,dy
\]

\[
C: \begin{align*}
\ dx &= dt & \ y &= 0 \\
\ dx &= dt & \ dy &= 0
\end{align*}
\]

\[
t \text{ starts at } 2 \text{ and ends at } -2
\]

\[
\int_C (2x + e^{-y})\,dx + (4y - xe^{-y})\,dy
\]

\[
= \int_{-2}^{2} (2t + 1)\,dt + 0 - \int_{-2}^{2} (2t + 1)\,dt
\]

\[
= - (t^2 + t)|_{-2}^{2} = -6 + 2 = -4
\]
Section 9.9 #20
Show that the given integral is independent of path and evaluate.
\[ \int_{(0,0,0)}^{(1,1,1)} (2x,3y^2,4z^3) \, dx + (0,0,0) \]

\[ F = \{2x,3y^2,4z^3\} \quad \text{curl} F = 0 \quad \Rightarrow \quad \text{F is a gradient field} \]

\[ \phi(x,y,z) = \int P \, dx = \int 2x \, dx = x^2 + G(y,z) \]
\[ \phi(x,y,z) = \int Q \, dy = \int 3y^2 \, dy = y^3 + H(x,z) \]
\[ \phi(x,y,z) = \int R \, dz = \int 4z^3 \, dz = z^4 + K(x,y) \]

\[ \Rightarrow x^2 + G(y,z) = y^3 + H(x,z) = z^4 + K(x,y) \]
\[ \Rightarrow H(x,z) = x^2 + z^4 + C_1, G(y,z) = y^3 + z^4 + C_2, \text{ and } K(x,y) = x^2 + y^3 + C_3 \]

\[ \phi(x,y) = x^2 + y^3 + z^4 + C \quad \Rightarrow \quad \int_{(0,0,0)}^{(1,1,1)} (2x,3y^2,4z^3) \, dz = \left( x^2 + y^3 + z^4 \right)_{(0,0,0)}^{(1,1,1)} = 3 \]

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Section 9.9 #26
Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \)
\[ \mathbf{F} = (2-e^x,2y-1,2-xe^x) \quad \mathbf{r} = (t,t^2,t^3) \text{ from } (-1,1,-1) \text{ to } (2,4,8) \]

\[ \text{curl} (\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2-e^x & 2y-1 & 2-xe^x \end{vmatrix} = \{0,-e^x-(e^x),0\} = 0 \quad \Rightarrow \quad \text{F is a gradient field} \]

\[ \phi(x,y,z) = \int P \, dx = \int (2-e^x) \, dx = 2x-xe^x + G(y,z) \]
\[ \phi(x,y,z) = \int Q \, dy = \int (2y-1) \, dy = y^2 - y + H(x,z) \]
\[ \phi(x,y,z) = \int R \, dz = \int (2-xe^x) \, dx = 2z-xe^x + K(x,y) \]

\[ \Rightarrow 2x-xe^x + G(y,z) = y^2 - y + H(x,z) = 2z-xe^x + K(x,y) \]
\[ G(y,z) = y^2 - y + 2z + C_1, H(x,z) = 2x-xe^x + 2z + C_2, \text{ and } K(x,y) = 2x + y^3 - y + C_3 \]
\[ \phi(x,y,z) = 2x-xe^x + y^2 - y + 2z + C \]

\[ \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \left[ (2x-xe^x+y^2-y+2z) \right]_{(-1,1,-1)}^{(2,4,8)} = (2e^2+16-2e^2) \]

\[ = (2e^2+16-2e^2) \]