12.6 The Fourier-Bessel Series

$x^2 y'' + xy' + \left( \alpha^2 x^2 - \nu^2 \right) y = 0$

The parametric Bessel equation of order $\nu$ has general solution on $(0, \infty)$ of

$y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$

$J_\nu(x)$ is called a Bessel function of the first kind of order $\nu$.

$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left( \frac{x}{2} \right)^{2n+\nu}$

$Y_\nu(x)$ is called a Bessel function of the second kind of order $\nu$.

For non-integer values of $\nu$

$Y_\nu(x) = \frac{\cos(\nu \pi) J_\nu(x) - J_{\nu+\pi}(x)}{\sin(\nu \pi)}$

For integer values (say $n$)

$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$

http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html
12.6 The Fourier-Bessel Series

Let $\nu = n \quad n = 0,1,2,…$

$Y_n(x)$: Bessel function of the second kind of order $n$.

http://mathworld.wolfram.com/BesselFunctionoftheSecondKind.html

12.6 The Fourier-Bessel Series

Let $\nu = n \quad n = 0,1,2,…$

The parametric Bessel differential equation becomes

$$x^2 y'' + xy' + (\alpha^2 x^2 - n^2) y = 0$$

with general solution on $(0, \infty)$ of

$$y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$$

The self-adjoint form of the differential equation is:

$$\frac{d}{dx}[xy'] + \left(\alpha^2 x - \frac{n^2}{x}\right)y = 0$$

we can identify $r(x) = x$, $q(x) = -\frac{n^2}{x}$, $p(x) = x$, and $\lambda = \alpha^2$

This is called a singular Sturm-Liouville problem when we add the following conditions:

$r(a) = 0$ and instead of 2 boundary conditions we only have

$A_1 y(b) + B_1 y'(b) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

For orthogonality, we need the solutions to be bounded at $x = a$. $r(0) = 0$ and we will only need

$A_2 y(b) + B_2 y'(b) = 0$

but $Y_n(0)$ is unbounded for all $n$, so for orthogonality we can only use $J_n(\alpha x)$.
12.6 The Fourier-Bessel Series

The self-adjoint form of the differential equation is:

$$\frac{d}{dx} \left[ xy' \right] + \left( \alpha^2 x - \frac{n^2}{x} \right) y = 0$$

we can identify $r(x) = x, q(x) = -\frac{n^2}{x}, p(x) = x,$ and $\lambda = \alpha^2$

with general solution

$$y = c_i J_n (\alpha x)$$

this gives a set of orthogonal functions

$$\{ J_n (\alpha x), J_m (\alpha x), \ldots \} \quad (\lambda_i = \alpha^2 \ i = 1, 2, \ldots)$$

that are orthogonal with respect to the weight function $p(x) = x$ on the interval $[0,b]$ but this is all contingent upon the $\alpha_i$

being defined by a boundary condition at $x = b$ of the type

$$A_i J_n (\alpha b) + B_i J'_n (\alpha b) = 0$$

by the chain rule

$$J'_n (\alpha x) = J_n (\alpha x) \frac{d}{dx} (\alpha x) = \alpha J'_n (\alpha x)$$

so the condition becomes:

$$A_i J_n (\alpha b) + B_i \alpha J'_n (\alpha b) = 0$$

We are interested in taking a function $f(x)$ and expanding it using Fourier eigenfunction expansion.

So now for $n = 0, 1, 2, \ldots$, we have the Bessel functions of order $n$

that will serve as our set of orthogonal functions used in the eigenfunction expansion of $f(x)$:

Let $n = 2$ for instance

$$\{ J_1 (\alpha x), J_2 (\alpha x), \ldots \}$$

is a set of orthogonal eigenfunctions

that are orthogonal with respect to the weight function $p(x) = x$ on the interval $[0,b]$ with corresponding eigenvalues $\lambda_i = \alpha^2 \ i = 1, 2, \ldots$

The expansion of $f(x)$ with Bessel functions $\{ J_n (\alpha x) \} \ i = 1, 2, \ldots$

is called a **Fourier – Bessel series.**

$$f(x) = \sum_{i=1}^{\infty} c_i J_n (\alpha_i x)$$

where $c_i = \frac{\int_{0}^{b} x J_n (\alpha x) f(x) dx}{\int_{0}^{b} (J_n (\alpha x))^2 dx}$
12.6 The Fourier-Bessel Series

In order to find the coefficients $c_i$, we need 3 properties of the Bessel $J$ function:

1. $J_n(-x) = (-1)^n J_n(x)$
2. $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$
3. $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

Three different versions of the boundary condition at $x = b$ lead to three different types of solutions

1. $J_n(ab) = 0$
2. $hJ_n(ab) + abJ'_n(ab) = 0$ \hspace{1cm} we’ll have 3 different results for $\int J_n(\alpha x)dx$
3. $J'_n(ab) = 0$

\[ f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \]

\[ c_i = \frac{2}{b^2 \left[ J_{n+1}(\alpha_i b) \right]^2} \int_0^b xJ_n(\alpha_i x) f(x) dx \]

when the $\alpha_i$ defined by the boundary condition $J_n(ab) = 0$

**example:**

#8 $f(x) = x^2$, \hspace{0.5cm} $0 < x < 1$

$J_2(\alpha) = 0$

\[ b = 1, n = 2, f(x) = x^2 \]

\[ c_i = \frac{2}{\left[ J_3(\alpha_i) \right]^2} \int_0^1 x^3 J_2(\alpha_i x) dx \]
12.6 The Fourier-Bessel Series

\[ f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \]

\[ c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) [J_n'(\alpha_i b)]^2} \int_0^b xJ_n(\alpha_i x) f(x) \, dx \]

when the \( \alpha_i \) are defined by

\[ hJ_n'(\alpha b) + \alpha b J_n'(\alpha b) = 0 \]

**Example:**

\#6 \( f(x) = 1, \quad 0 < x < 2 \)

\( J_0(2\alpha) + \alpha J'_0(2\alpha) = 0 \)

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12.6 The Fourier-Bessel Series

\[ f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x) \quad \quad c_1 = \frac{2}{b^2} \int_0^b x f(x) \, dx \]

\[ c_i = \frac{2}{b^2 [J_0'(\alpha b)]^2} \int_0^b xJ_0(\alpha_i x) f(x) \, dx \]

when the \( \alpha_i \) are defined by

the boundary condition \( J'_0(\alpha b) = 0 \)

**Example:**

\#4 \( f(x) = 1, \quad 0 < x < 2 \)

\( J'_0(2\alpha) = 0 \)
12.6 The Fourier-Bessel Series

\[ f(x) = x^2, \quad 0 < x < 1 \]

\[ J_2(\alpha) = 0 \]

\[ c_i = \frac{2}{[J_3(\alpha_i)]^2} \int_0^1 x^3 J_2(\alpha_i x) \, dx \]

\[ x = 0 \Rightarrow t = 0 \]

\[ \frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \]

\[ \Rightarrow \frac{d}{dt}[t^n J_n(t)] = t^n J_{n-1}(t) \]

\[ c_i = \frac{2}{\alpha_i^4 [J_3(\alpha_i)]^2} \int_0^{\alpha_i} t^3 J_2(t) \, dt \]

\[ c_i = \frac{2}{\alpha_i^4 [J_3(\alpha_i)]^2} \int_0^{\alpha_i} \frac{d}{dt}[t^3 J_3(t)] \, dt \]

\[ c_i = \frac{2}{\alpha_i^4 [J_3(\alpha_i)]^2} \left[ t^3 J_3(t) \right]_0^{\alpha_i} = \frac{2\alpha_i J_3(\alpha_i)}{\alpha_i^4 [J_3(\alpha_i)]^2} \]

\[ f(x) = 2 \sum_{i=1}^{\infty} \frac{1}{\alpha_i J_3(\alpha_i)} J_2(\alpha_i x) \]