13.2 Classical PDE’s and Boundary Value Problems

<table>
<thead>
<tr>
<th>Heat Equation</th>
<th>Wave Equation</th>
<th>Laplace’s Equation</th>
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<tbody>
<tr>
<td>Solution: $u(x,t)$</td>
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</tr>
<tr>
<td>$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ $k &gt; 0$</td>
<td>$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ $\alpha &gt; 0$</td>
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<tr>
<td>or $k u_{xx} = u_t$ $k &gt; 0$</td>
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- ________ (transfer) by ________ in a rod or thin wire
  - $u = \quad x = \quad t =$
- ________ (financial mathematics) reduces to heat equation
  - measures mechanical vibrations of a ________
  - ________ and ________ satisfy the wave eq. in a long cable (telegraph eq.)
  - ________ equation (financial mathematics)
  - fluid mechanics
  - acoustics
  - elasticity

- time independent
  - ________ temperature distribution throughout a thin ________
  - ________ and ________ other areas:
  - electrostatic potential
  - gravitational potential
  - velocity in fluid mechanics

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We need to consider ________ conditions and ________ values:

**Initial conditions**: when time $t = 0$ $\Rightarrow$

Heat equation:

- $u(x, 0) = f(x)$
  - (Initial temp. distribution)

Wave equation:

- can also specify initial velocity of the string
  - $g(x) = 0$ $\Rightarrow$
  - $u = 0$ at $x = 0$
  - $u = 0$ at $x = L$
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**Boundary conditions**: 3 types:

- \( u = 0 \) : Condition on the function \( u \) at the endpoints
  - **Heat equation**: temperature at the left and right ends of the rod
  - **Wave equation**: ends of the string are fixed to the \( x \)-axis could also move in a transverse manner according to a function of time
  - **Laplace's equation**: temperature on the boundary of the plate

\[ u = 0 \]

- \( u = f(x) \): Condition on the normal derivative of the function \( u \) (directional derivative of \( u \) perpendicular to the boundary) at the endpoints
  - **Heat equation**: no temperature change at the left or right ends of the rod \( \Rightarrow \) **end is** \( u = 0 \)
  - **Wave equation**: string endpoint which is free to move in a transverse direction
  - **Laplace's equation**: no temperature change through the side of the plate
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- _______: Condition that is a combination of Dirichlet and Neumann, mainly used for the heat equation

The heat loss or heat gain represented as ________ through an endpoint.

____________________ states that this is proportional to the difference between that temperature at the boundary and the temperature of the surrounding medium.

\[ h > 0, h \text{ and } u_m \text{ are constants} \]

Assume the rod is at a higher temperature than the medium surrounding the ends:

⇒

This explains the heat gain on the left end and the heat loss on the right end.

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Heat transfer through the lateral surface ⇒

The pde becomes:

![Diagram of a rod with insulated ends and heat transfer through the lateral surface]
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**Set up the boundary value problem:**

1. Heat equation - a rod of length $L$
   - The left end is held at temperature zero, and the right end is insulated. The initial temperature is $f(x)$ throughout.
   - Solve:
   - Subject to:

2. Heat equation - a rod of length $L$
   - The left end is held at temperature $100^\circ$, and there is heat transfer from the right end into the surrounding medium at temperature zero. The initial temperature is $f(x)$ throughout.
   - Solve:
   - Subject to:

3. Heat equation - a rod of length $L$
   - The left end is held at temperature zero, and the right end is insulated. The initial temperature is $f(x)$ throughout.
   - Solve:
   - Subject to:

10. Wave equation - a string of length $L$
    - The ends are secured to the $x$–axis, and the string is initially at rest on that axis. An external vertical force proportional to the horizontal distance from the left end acts on the string for $t > 0$.
    - Solve:
    - Subject to: