**Section 17.3**

Sets in the Complex Plane

The points \( z = x + iy \) that lie on the _____ of radius ____ centered at the point ____.

\[ |z - z_0| = \rho \]

The points \( z = x + iy \) that lie within (_______) the circle of radius \( \rho \) centered at the point \( z_0 \).

This is called a ___________ of \( z_0 \) or an __________.

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Take a point \( z_0 \) of a set \( S \) of the complex plane. If there exists some neighborhood of \( z_0 \) that lies entirely within \( S \), then \( z_0 \) is called an __________ of \( S \).

\( S \) is called an _______ if every point \( z \) of the set \( S \) is an _______ point.

Examples:
If _______ neighborhood of a point \( z_0 \) contains at least one point that is in a set \( S \) and at least one point that is not in \( S \), then \( z_0 \) is called a _____________ of \( S \).

The set of _____ boundary points of \( S \) is called the __________ of \( S \).

The _________ of an open set is a ________.

A closed set contains its ________ (all of its ____________)

Take an _________ \( S \).

If any pair of points in \( S \) can be connected by a _____ number of line segments, then \( S \) is called _________.

An open connected set is also called a __________.

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**Section 17.4**

**Complex Functions**

Let \( S \) be a subset of \( \mathbb{C} \).

A **complex function** \( f \) is a rule that assigns a complex number \( w \) to every complex number \( z \) in \( S \).
\[ z = x + iy \quad \text{and} \quad w = u(x, y) + iv(x, y) \]

Image of z

\[ w = f(z) = f(x + iy) = u(x, y) + iv(x, y) \]

**Example:**

\[ f(z) = z + (\overline{z})^2 \]

\[ f(1 + 2i) = \quad f\left(\frac{1}{2} - i\right) = \]

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**Limit of a complex function**

If for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ |f(z) - L| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta, \]

then we say that the limit of \( f(x) \) as \( z \) approaches \( z_0 \) exists and is equal to \( L \).

In symbols, \( \lim_{z \to z_0} f(z) = L \).

For any given \( \varepsilon \)-neighborhood of \( L \) containing the image of \( z \) (called \( f(z) \)), one can always find a \( \delta \)-neighborhood of \( z_0 \) containing \( z \) such that the images of all the points in the \( \delta \)-neighborhood of \( z_0 \) lie within the \( \varepsilon \)-neighborhood about \( L \).

Like in the multivariable limit of Calc II, \( z \to z_0 \) from _____ direction.
Definitions:

\( f(z) \) is ______________ if \( \lim_{z \to z_0} f(x) = f(z_0) \).

\( f(z) \) is ______________ if it is continuous at every point in the domain.

The __________ of \( f \) at a point \( z_0 \) is

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

(provided this limit exists)

If the limit exists, \( f \) is said to be ______________ at \( z_0 \).

If \( f \) is differentiable at \( z_0 \) and at every point in some neighborhood of \( z_0 \), then \( f \) is called ______________.

\( f(z) \) is called ______________ (or just __________) if it is analytic at every point in the domain.

If \( f(z) \) is analytic on the ______ complex plane, then \( f \) is called ________.

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Let \( f(z) = \overline{z} \). Find \( f'(z) \) if it exists.

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

\[
z = f(z) = \Delta z = f(z + \Delta z) =
\]

\[
f(z + \Delta z) - f(z) = (x + \Delta x) - i(y + \Delta y) - [x - iy] =
\]

\[
f'(z) = \lim_{\Delta z \to 0}
\]

a) let \( \Delta z \to 0 \) along a ________ line, (\( \Rightarrow \Delta y = 0 \))

b) let \( \Delta z \to 0 \) along a ________ line, (\( \Rightarrow \Delta x = 0 \))

\[
\lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta z}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta z}{\Delta z}
\]

\( \Rightarrow \) the limit ________________, \( f(z) = \overline{z} \) is __________ differentiable.
The differentiation rules for complex functions are the same as for real-valued functions.

**Constant Rules**
\[
\frac{d}{dz}(c) = 0 \quad \text{and} \quad \frac{d}{dz}(cf(z)) = cf'(z)
\]

**Quotient Rule**
\[
\frac{d}{dz} \left( \frac{f(z)}{g(z)} \right) = \frac{f'(z)g(z) - g'(z)f(z)}{(g(z))^2}
\]

**Sum/Difference Rules**
\[
\frac{d}{dz}(f(z) \pm g(z)) = f'(z) \pm g'(z)
\]

**Product Rule**
\[
\frac{d}{dz}(f(z) \cdot g(z)) = f'(z)g(z) + g'(z)f(z)
\]

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Solve \( z^2 - 2z + 2 = 0 \)  
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\( z = x + iy \), \( x \) and \( y \) real

\[(x + iy)^2 - 2(x + iy) + 2 = 0\]

\[
( ) + ( )i = 0
\]

\( = 0 \quad \text{and} \quad = 0 \)

\( 2xy - 2y = 0 \Rightarrow 2y(x - 1) = 0 \quad \text{either} \)

\( y = 0 \Rightarrow x^2 - y^2 - 2x + 2 = 0 \) becomes

\( x^2 - 2x + 2 = 0 \) which has no solution for real \( x \)

\( x = 1 \Rightarrow x^2 - y^2 - 2x + 2 = 0 \) becomes

\( 1 - y^2 - 2 + 2 = 0 \Rightarrow \)

Roots of \( z^2 - 2z + 2 = 0 \) are \( z = \) and \( z = \)

So \( z^2 - 2z + 2 = [ ] \)
Solve \( z^2 - 2i = 0 \)

\( z = x + iy, \) \( x \) and \( y \) real

\((x + iy)^2 - 2i = 0\)

\((\quad) + (\quad)i = 0\)

\(= 0\) and \(= 0\)

\(2xy - 2 = 0 \Rightarrow 2(xy - 1) = 0 \Rightarrow\)

\(\Rightarrow\)

\(y = \frac{1}{x} \Rightarrow x^2 - y^2 = 0\) becomes

\(x^2 - \frac{1}{x^2} = 0 \Rightarrow x^4 = 1\) so

Roots of \( z^2 - 2i \)

are \( z = \) and \( z = \)

So \( z^2 - 2i = [\quad] \)

\[ \lim_{z \to 1+i} \frac{z^2 - 2z + 2}{z^2 - 2i} \]

\[ = \lim_{z \to 1+i} \frac{z - (1 + i)(z - (1 - i))}{z - (1 + i)(z - (-1 - i))} \]

\[ = \lim_{z \to 1+i} \frac{(1 + i) - (1 - i)}{(1 + i) - (-1 - i)} = \]

\[ = \frac{2i(\quad)}{(2 + 2i)(\quad)} = \frac{8}{8} = \]