Section 17.5 Cauchy-Riemann Equations

If \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is differentiable at \( z \), then the partial derivatives of \( u \) and \( v \) exist at the point \( z \) and satisfy the Cauchy–Riemann equations \( u_x = v_y \) and \( u_y = -v_x \) at \( z \).

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]
where:
- \( \Delta z = \Delta x + i\Delta y \)
- \( f(z) = u(x, y) + iv(x, y) \)
- \( f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) \)

\[
f'(z) = \lim_{\Delta x + i\Delta y \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y}
\]

\[
f'(z) = \lim_{\Delta x + i\Delta y \to 0} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + i \lim_{\Delta x + i\Delta y \to 0} \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y}
\]

These limits have to exist as \( \Delta x + i\Delta y \to 0 \) from any direction.

Consider two paths; horizontally and vertically.

a) as \( \Delta x + i\Delta y \to 0 \) horizontally \( \Delta y = 0 \), thus the limits become:

\[
f'(z) = \left( \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \right) + i \left( \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right)
\]

\[
f'(z) = u_x + iv_x
\]

b) as \( \Delta x + i\Delta y \to 0 \) vertically \( \Delta x = 0 \), thus the limits become:

\[
f'(z) = \left( \lim_{i\Delta y \to 0} \frac{u(x, y + i\Delta y) - u(x, y)}{i\Delta y} \right) + i \left( \lim_{i\Delta y \to 0} \frac{v(x, y + i\Delta y) - v(x, y)}{i\Delta y} \right)
\]

\[
f'(z) = \frac{1}{i} \left( \text{partial derivative of } u \text{ w.r.t. } y \right) + i \left( \text{partial derivative of } v \text{ w.r.t. } y \right)
\]

\[
f'(z) = v_y - iu_y
\]

Equating real parts: \( u_x = v_y \)  
Equating imaginary parts: \( u_y = -v_x \)
We just saw:
If \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is differentiable at \( z \),
then the partial derivatives of \( u \) and \( v \) exist at the point \( z \) and
satisfy the Cauchy–Riemann equations \( u_x = v_y \) and \( u_y = -v_x \) at \( z \).

The converse is not necessarily true. It is possible for \( u \) and \( v \) to satisfy
the Cauchy–Riemann equations at \( z \) and for \( f \) to not be differentiable at \( z \).

Example: \( f(z) = \begin{cases} \frac{z^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases} \)

\[ \frac{\Delta u}{\Delta x} = \frac{0 - 0}{\Delta x} = 1 \]
\[ \frac{\Delta v}{\Delta y} = 0 \]

\[ u_x(0, 0) = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \frac{(\Delta x)^2 - 0}{(\Delta x)^2 + 0} = 1 \]
\[ v_y(0, 0) = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = 0 \]

So \( u_x(0, 0) = 0 = v_y(0, 0) \),
so \( u_x = -v_y \).

From last lecture we saw:
If \( f \) is differentiable at \( z_0 \) and at every point in some neighborhood of \( z_0 \),
then \( f \) is called analytic at \( z_0 \).

\( f(z) \) is called analytic on a domain (or just analytic) if it
is analytic at every point in the domain.

So now we can say,
\( f(z) \) is analytic on a domain \( \Rightarrow \) the partial derivatives of \( u \) and \( v \) exist
and satisfy the Cauchy-Riemann equations on the entire domain.

The Cauchy-Riemann equations by themselves are not enough to ensure the
analyticity of a function. We need to add a couple of conditions:

If two real- valued functions \( u(x, y) \) and \( v(x, y) \)
a) are continuous and have continuous first- order partial derivatives in a domain \( D \) and
b) satisfy the Cauchy - Riemann equations on the entire domain,
then \( f(z) = u(x, y) + iv(x, y) \) is analytic in \( D \).

We can find \( f'(z) \) by:

a) \( f'(z) = u_x + iv_x \)
b) \( f'(z) = v_y - iu_y \)
Solutions of Laplace’s equation \( u_{xx} + u_{yy} = 0 \) that have continuous second order partial derivatives in a domain \( D \), are called harmonic in \( D \).

\[
f(z) = u(x, y) + iv(x, y) \Rightarrow u(x, y) \text{ and } v(x, y)
\]
is analytic in \( D \) are harmonic in \( D \)

**Proof:** \( f \) analytic \( \Rightarrow \) 1) \( u_x = v_y \) and 2) \( u_y = -v_x \).

- Differentiate 1) w.r.t. \( x \) \( \Rightarrow u_{xx} = v_{yx} \)
- Differentiate 2) w.r.t. \( y \) \( \Rightarrow u_{yy} = -v_{xy} \)
- Adding gives: \( u_{xx} + u_{yy} = v_{yx} - v_{xy} \) \( \Rightarrow u_{xx} + u_{yy} = 0 \), thus \( u \) is harmonic.

- Differentiate 1) w.r.t. \( y \) \( \Rightarrow u_{xy} = v_{yy} \)
- Differentiate 2) w.r.t. \( x \) \( \Rightarrow u_{yx} = -v_{xx} \)
- Subtracting gives: \( v_{xx} + v_{yy} = u_{xy} - u_{yx} \) \( \Rightarrow v_{xx} + v_{yy} = 0 \), thus \( v \) is harmonic.

Given a \( u(x, y) \) that is harmonic in a domain \( D \), we can find a \( v(x, y) \) called the conjugate harmonic function of \( u \) so that \( f(z) = u(x, y) + iv(x, y) \) is analytic in \( D \).

**Example:** \( u(x, y) = 4x y^3 - 4x^3 y + x \)

\[
u_x = 4y^3 - 12x^2 y + 1 \text{ and } u_y = 12x y^2 - 4x^3
\]

- \( C - R \) equations: \( u_x = v_y \) and \( u_y = -v_x \)

\[
\Rightarrow v_y = 4y^3 - 12x^2 y + 1
\]

\[
\Rightarrow \int v_y \, dy = \int \left( 4y^3 - 12x^2 y + 1 \right) dy
\]

\[
\Rightarrow v(x, y) = y^4 - 6x^2 y^2 + y + K(x)
\]

so \( v_x = -12x y^2 + K'(x) \)

\[
u_y = -v_x \Rightarrow 12y^2 - 4x^3 = -12x y^2 + K'(x) \Rightarrow K'(x) = 4x^3
\]

\[
\Rightarrow K(x) = \int K'(x) \, dx = \int 4x^3 \, dx \Rightarrow K(x) = x^4 + C
\]

\[
:\therefore v(x, y) = y^4 - 6x^2 y^2 + y + x^4 + C
\]

\[
f(z) = (4x y^3 - 4x^3 y + x) + i \left( y^4 - 6x^2 y^2 + y + x^4 + C \right) \text{ is analytic}
\]