Section 18.2 Cauchy-Goursat Theorem

Simply connected domain
- every simple closed contour lying entirely in the domain can be shrunk to a point without leaving D
- "no holes"

Multiply connected domain
- connected but not simply connected
- one or more "holes"

Cauchy-Goursat Theorem
Let \( f(z) \) be analytic in a simply connected domain \( D \).
For every simple closed contour \( C \) in \( D \), \( \oint_C f(z) dz = 0 \).

Example:
\[
\oint_C \frac{z-3}{z^2 - 2z + 2} \, dz
\]
\( C \) is the unit circle \(|z|=1\)

\[
= \oint_C \frac{z-3}{z-(1+i)} \frac{z-3}{z-(1-i)} \, dz
\]
Analytic inside and on the unit circle

\[
\Rightarrow \oint_C \frac{z-3}{z^2 - 2z + 2} \, dz = 0
\]
What about for multiply connected domains?

Introduce a cut from A to B

This effectively switches the orientation of the inner contour

\[
\oint_{C} f(z)\,dz + \oint_{C_1} f(z)\,dz = 0
\]

\[
\oint_{C} f(z)\,dz = -\oint_{C_1} f(z)\,dz
\]

\[
\oint_{C} f(z)\,dz = \oint_{C_1} f(z)\,dz
\]

Deformation of contours

\[
\oint_{C} \frac{1}{z - i}\,dz
\]

C is the outer contour below

Deform contour C into a more convenient circular contour C₁

\[C_1\text{ parametrized by } z = i + e^{it} \quad 0 \leq t \leq 2\pi\]

\[
\oint_{C} f(z)\,dz = \int_{a}^{b} f(z(t))z'(t)\,dt
\]

\[
f(z(t)) = \frac{1}{i + e^{it} - i} = e^{-it} \quad z'(t) = ie^{it}
\]

\[
f(z(t))z'(t) = e^{-it} \cdot ie^{it} = i
\]

\[
\oint_{C} \frac{1}{z - i}\,dz = \int_{0}^{2\pi} i\,dt = 2\pi i
\]
Let $z_0$ be in the interior of any simple closed contour $C$.

\[ \oint_C \frac{1}{(z-z_0)^n} \, dz = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases} \]

For $n = 1$:

- Deform to a convenient circular contour $C_r$ defined by $|z-z_0|=r$.
- $r$ small enough so that $C_r$ lies entirely inside $C$.

\[ z = z_0 + re^{it} \quad z'(t) = ire^{it} \]

\[ f(z(t)) = \frac{1}{z_0 + re^{it} - z_0} = \frac{1}{r} e^{-it} \]

\[ f(z(t)) z'(t) = i \]

\[ \oint_C \frac{1}{(z-z_0)^n} \, dz = \int_0^{2\pi} i \, dt = 2\pi i \]

\[ \int_0^{2\pi} i \, dt = 2\pi i \]

\[ \int_0^{2\pi} e^{(1-n)i} \, dt = \int_0^{2\pi} \frac{e^{(1-n)i}}{(1-n)i} \, dt \]

\[ = \left[ e^{(1-n)i} \right]_0^{2\pi} = e^{2\pi(1-n)i} - 1 \]

\[ = 0 \quad (\text{since } e^{2\pi(1-n)i} = 1) \]

$$\int_{C_r} \frac{-3z+2}{z^2-8z+12} \, dz = -4\int_{C_r} \frac{1}{z-6} \, dz + \frac{1}{z-2} \, dz$$

**C:** $|z|=9$

"hole" at 6  "hole" at 2

\[ \frac{-3z+2}{z-6}(z-2) = \frac{A}{z-6} + \frac{B}{z-2} = \frac{-4}{z-6} + \frac{1}{z-2} \]

$z=6:$ $A = \frac{-3(6)+2}{6-2} = \frac{-16}{4} = -4$

$z=2:$ $B = \frac{-3(2)+2}{2-6} = \frac{-4}{-4} = 1$

\[ = -4(2\pi) + (2\pi) \]

\[ = -6\pi i \]