Counting Combinations

Order doesn't matter

\( n = \) the total number of objects you are choosing from

\( r = \) the number of objects you are choosing

\( C_{n,r} = \) total number of ways to choose \( r \) different objects out of a total of \( n \) when order doesn't matter.

\[
C_{n,r} = \frac{n!}{r!(n-r)!}
\]

Example: Total number of 5 card hands that can be dealt from a standard 52 card deck

\[
C_{52,5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot \ldots \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 47 \cdot 46 \cdot 45 \cdot \ldots \cdot 2 \cdot 1} = 2,598,960
\]

On TI-83:
Type 52 first

To PRB

Now type 5

\[
\begin{array}{c}
52! \\
\text{MATH} \\
\text{NUM} \\
\text{CPX} \\
\text{PRB} \\
\frac{52!}{5! \cdot 47!} \\
\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot \ldots \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 47 \cdot 46 \cdot 45 \cdot \ldots \cdot 2 \cdot 1} \\
2,598,960
\end{array}
\]
Counting Poker Hands and Finding Probabilities
by N. Rimmer

52 Card deck

26 Black Cards
13 Spades
13 Clubs

26 Red Cards
13 Hearts
13 Diamonds

These are called suits

In each suit, you have the following 13 cards:

A K Q J 10 9 8 7 6 5 4 3 2
ace king queen jack

This is called rank of the card

rank: 10
suit: spades
card: 10 of spades
How many ways are there to get 5 cards in rank order?

A K Q J 10  
K Q J 10 9  
Q J 10 9 8  
J 10 9 8 7  
10 9 8 7 6

9 8 7 6 5  
8 7 6 5 4  
7 6 5 4 3  
6 5 4 3 2  
5 4 3 2 A

This is called a straight. (Here we are not considering suits)
**Royal Flush** - Ace, King, Queen, Jack, 10 all of the same suit

![Royal Flush Cards](image)

One Royal Flush for each suit

\[
P(\text{Royal Flush}) = \frac{\binom{4}{1}}{\binom{52}{5}}
\]

\[
P(\text{Royal Flush}) = \frac{4}{2,598,960} \approx 0.0000015391
\]

**Odds** (Royal Flush) \( \approx 1 \) in 649,740
**Straight Flush** - Five cards in rank order of the same suit

\[
P(\text{Straight Flush}) = \frac{\binom{4}{1} \cdot \binom{10}{1} - 4}{\binom{52}{5}}
\]

\[
P(\text{Straight Flush}) = \frac{36}{2,598,960} \approx 0.0000138517
\]

**Odds** (Straight Flush) \( \approx 1 \) in 72,193
4 of a kind - Four cards of one rank and one other card

\[
P(\text{4 of a kind}) = \frac{C_{13,1} \cdot C_{4,4} \cdot C_{48,1}}{C_{52,5}}
\]

\[
P(\text{4 of a kind}) = \frac{624}{2,598,960} \approx 0.0002400960
\]

Odds (4 of a kind) \approx 1 \text{ in } 4,165
Full House - Three cards of one rank and two cards of another rank

\[
P(\text{Full House}) = \frac{\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2}}{\binom{52}{5}}
\]

\[
P(\text{Full House}) = \frac{3,744}{2,598,960} \approx 0.0014405762
\]

Odds \( \text{(Full House)} \) \( \approx \) 1 in 694
Flush - Five cards of the same suit

\[ P(\text{Flush}) = \frac{\binom{4}{1} \cdot \binom{13}{5} - 4}{\binom{52}{5}} \]

\[ P(\text{Flush}) = \frac{5,108}{2,598,960} \approx 0.0019654015 \]

Odds (Flush) \approx 1 \text{ in } 509
**Straight** - Five consecutive cards each of any suit

\[
P(\text{Straight}) = \frac{C_{10,1} \cdot C_{4,1} \cdot C_{4,1} \cdot C_{4,1} \cdot C_{4,1}}{C_{52,5}} - \frac{4}{36} \quad \text{royal straight flushes}
\]

\[
P(\text{Straight}) = \frac{10,200}{2,598,960} \approx 0.0039246468
\]

\[
\text{Odds (Straight)} \approx 1 \text{ in } 255
\]
3 of a Kind – Three cards of one rank and two cards of different rank

\[ P(3 \text{ of a kind}) = \binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{52}{5} \]

\[ P(3 \text{ of a kind}) = \frac{54,912}{2,598,960} \approx 0.0211284514 \]

Odds (3 of a kind) \( \approx 1 \text{ in 47} \)
2 pair - 2 cards of one rank, 2 cards of another rank and another card of a third rank

\[ P(2 \text{ pair}) = \frac{C_{13,2} \cdot C_{4,2} \cdot C_{4,2} \cdot C_{44,1}}{C_{52,5}} \]

\[ P(2 \text{ pair}) = \frac{123,552}{2,598,960} \approx 0.0475390156 \]

Odds (2 pair) \( \approx 1 \) in 21
One pair - 2 cards of one rank, 3 other cards of different rank

\[
P(\text{One pair}) = \frac{\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}}{\binom{52}{5}}
\]

\[
P(\text{One pair}) = \frac{1,098,240}{2,598,960} \approx 0.4225690276
\]

\[
\text{Odds (One pair)} \approx 2 \text{ in 5}
\]
No pair – 5 different ranks each of any suit

\[ P(\text{No pair}) = \frac{\binom{13}{5} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}}{\binom{52}{5}} \]

\[ P(\text{No pair}) = \frac{1,302,540}{2,598,960} \approx 0.5011773940 \]

Odds (No pair) \(\approx\) 1 in 2
### 5-Card Poker Hand Summary

<table>
<thead>
<tr>
<th>Hand</th>
<th>How many</th>
<th>Probability</th>
<th>Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Royal Flush</td>
<td>4</td>
<td>0.000001539</td>
<td>1 in 649,740</td>
</tr>
<tr>
<td>Straight Flush</td>
<td>36</td>
<td>0.000013857</td>
<td>1 in 72,193</td>
</tr>
<tr>
<td>4 of a Kind</td>
<td>624</td>
<td>0.000240096</td>
<td>1 in 4,165</td>
</tr>
<tr>
<td>Full House</td>
<td>3,744</td>
<td>0.001440576</td>
<td>1 in 694</td>
</tr>
<tr>
<td>Flush</td>
<td>5,108</td>
<td>0.001965402</td>
<td>1 in 509</td>
</tr>
<tr>
<td>Straight</td>
<td>10,200</td>
<td>0.003924647</td>
<td>1 in 255</td>
</tr>
<tr>
<td>3 of a Kind</td>
<td>54,912</td>
<td>0.021128451</td>
<td>1 in 47</td>
</tr>
<tr>
<td>Two Pair</td>
<td>123,552</td>
<td>0.047539016</td>
<td>1 in 21</td>
</tr>
<tr>
<td>One Pair</td>
<td>1,098,240</td>
<td>0.422569028</td>
<td>2 in 5</td>
</tr>
<tr>
<td>No Pair</td>
<td>1,302,540</td>
<td>0.501177394</td>
<td>1 in 2</td>
</tr>
</tbody>
</table>

Total: 2,598,960