Exponential of a real \((3 \times 3)\)-matrix with repeated eigenvalues

If \(A\) is a real \((3 \times 3)\)-matrix with real eigenvalues \((\lambda_1, \lambda_2, \lambda_2)\) so the characteristic equation
\[
p(\lambda) = \det(A - \lambda I) = 0
\]
has a single real root \(\lambda_1\) and a double real root \(\lambda_2\) then the exponential of \(A\) is given by
\[
et A = e^{\lambda_2 t} I + te^{\lambda_2 t}(A - \lambda_2 I) \\
+ (\lambda_1 - \lambda_2)^{-2}(e^{\lambda_1 t} - e^{\lambda_2 t} -(\lambda_1 - \lambda_2)te^{\lambda_2 t})(A - \lambda_2 I)^2
\]

If \(A\) is a real \((3 \times 3)\)-matrix with one real eigenvalue \((\lambda, \lambda, \lambda)\) so \(\lambda\) is a triple root of the characteristic equation
\[
p(\lambda) = \det(A - \lambda I) = 0
\]
then the exponential of \(A\) is given by
\[
et A = e^{\lambda t}(I + t(A - \lambda I) + \frac{1}{2}t^2(A - \lambda I)^2).
\]
Green’s Functions

Here are the Green’s functions for second order constant coefficient linear differential equation. We find a particular solution \( y_p \) to the differential equation \( Ly = F \) where \( L = P(D) = (D-\lambda_1)(D-\lambda_2) \). Note \( y_p(0) = 0 \) and \( y_p'(0) = 0 \).

Case 1. Two distinct real roots \( \lambda_1 \) and \( \lambda_2 \).

\[
y_p(x) = \frac{1}{\lambda_1 - \lambda_2} \int_0^x (e^{\lambda_1 (x-t)} - e^{\lambda_2 (x-t)}) F(t) \, dt
\]

\[
y_p(x) = \frac{e^{\lambda_1 x}}{\lambda_1 - \lambda_2} \int_0^x e^{-\lambda_1 t} F(t) \, dt - \frac{e^{\lambda_2 x}}{\lambda_1 - \lambda_2} \int_0^x e^{-\lambda_2 t} F(t) \, dt
\]

Case 2. One real repeated root \( \lambda \) so \( Ly = P(D)y = (D-\lambda)^2 y \).

\[
y_p(x) = \int_0^x e^{\lambda(x-t)} (x-t) F(t) \, dt
\]

\[
y_p(x) = x e^{\lambda x} \int_0^x e^{-\lambda t} F(t) \, dt - e^{\lambda x} \int_0^x t e^{-\lambda t} F(t) \, dt
\]

Case 3. \( Ly = P(D) = (D-a)^2 + b^2 = D^2 - 2aD + (a^2+b^2) \), \( \lambda = a \pm ib \)

\[
y_p(x) = \frac{1}{b} \int_0^x e^{a(x-t)} (\sin(bt)\cos(bt) - \cos(bt)\sin(bt)) F(t) \, dt
\]

\[
y_p(x) = \frac{1}{b} e^{ax} \sin(bx) \int_0^x e^{-at} \cos(bt) F(t) \, dt
\] - \[
\frac{1}{b} e^{ax} \cos(bx) \int_0^x e^{-at} \sin(bt) F(t) \, dt
\]

Using a trigonometric identity for \( \sin(x-y) \) this can be written as

\[
y_p(x) = \frac{1}{b} \int_0^x e^{a(x-t)} \sin(b(x-t)) F(t) \, dt
\]