Last time. We want to solve the equation

$$\frac{d\vec{x}}{dt}(t) = A\vec{x}(t).$$

Where A is a given matrix that does not depend on (t).

Eigenvalues and eigenvectors

Suppose A is an (n x n)-matrix. We say λ is an eigenvalue of A if there is a non zero vector $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \lambda\vec{x}$.
Suppose A is a (3 x 3)-matrix and it has three eigenvalue λ_1, λ_2, λ_3 and three eigenvectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$. Let us suppose the eigenvectors are linearly independent. Then each vector $\vec{w}$ is a linear combination of the $\vec{v}$'s. In fact

$$\vec{w} = s_1\vec{v_1} + s_2\vec{v_2} + s_3\vec{v_3}$$

$$A^n\vec{w} = s_1\lambda_1^n\vec{v_1} + s_2\lambda_2^n\,\vec{v_2} + s_3\lambda_3^n\vec{v_3}$$

So in terms of the $\vec{v}$'s the matrix A is diagonal so

$$e^{tA} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix} \qquad \text{in terms of the } \vec{v}$$

Now the traditional basis for $\mathbb{R}^3$ is

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and to compute e^{tA} we have the basis $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$. We form the change of basis matrix S. Given a vector

$$\vec{w} = s_1\vec{v_1} + s_2\vec{v_2} + s_3\vec{v_3}$$

to express $\vec{w}$ in terms of the standard basis we use the matrix

$$S = [\vec{v_1}, \vec{v_2}, \vec{v_3}] = \begin{bmatrix} (\vec{v_1})_1 & (\vec{v_2})_1 & (\vec{v_3})_1 \\ (\vec{v_1})_2 & (\vec{v_2})_2 & (\vec{v_3})_2 \\ (\vec{v_1})_3 & (\vec{v_2})_3 & (\vec{v_3})_3 \end{bmatrix}$$

$$S \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = w_x \vec{\imath} + w_y \vec{\jmath} + w_z \vec{k}$$

To express a vector in standard coordinates (w_x, w_y, w_z) in terms of the $\vec{v}$'s we use the inverse matrix S^{-1}

$$S^{-1} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Then

$$e^{tA} = S \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix} S^{-1}$$

$$\text{note} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = S^{-1} A S$$

A matrix is diagonalizable if $A = SDS^{-1}$

A is diagonalizable if the set of eigenvectors span $\mathbb{R}^n$

How to find the eigenvalues

$$p(\lambda) = \det(A - \lambda I).$$

$$p(\lambda) = (-\lambda)^n + tr(A)\ (-\lambda)^{n-1} + \cdots + \det(A)$$

$$\text{trace } A = \sum_{k=1}^{n} a_{ii}$$

characteristic polynomial
eigenvalues are the roots to the equation $p(\lambda) = 0$.

If there are n distinct roots then the eigenvector form a basis for $\mathbb{R}^n$. It there are less than n distinct roots the eigenvectors may still form a basis. You have to check.
It the eigenvectors do not form a basis then A is said to be defective.
Lets do a problem.

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$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

They say the characteristic polynomial is $(\lambda-1)^2(\lambda-3)$

Note the book defines

$$p(\lambda) = \det(\lambda I - A) = -(\lambda-1)^2(\lambda-3)$$

You should be aware that other authors define $p(\lambda) = \det(\lambda I-A)$.
For the A above we have

$$\det(A - \lambda I) = (2-\lambda)^2(1-\lambda) -(1-\lambda) = (1-\lambda)(4-4\lambda+\lambda^2-1)$$
$$= (1-\lambda)(3-4\lambda+\lambda^2) = (1-\lambda)(1-\lambda)(3-\lambda)$$

Find the eigenvector for $\lambda = 3$.

$$A - 3I = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$$(A-3I)\vec{v} = 0 \qquad \vec{v} = (x,y,z)$$

call the components anything you want

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

middle equation says $-3y = 0$ so $y = 0$

Top and bottom equations say

$$-x + z = 0$$

$$x - z = 0$$

Pick the most simple solution $\vec{v} = (1,0,1)$

What about the eigenvalue $\lambda = 1$.

$$(A - I)\vec{v} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

middle equation tell you nothing
top and bottom tell you the same thing

so there is only one equation

$$x - y + z = 0$$

Pick two solution. How to pick. Anyway you want.

You want integer coefficients.

I like (1,1,0) and (0,1,1)

So we have three eigenvectors.

$$\vec{v_1}\ \vec{v_2}\ \vec{v_3} = \begin{array}{c} \begin{matrix} 3 & 1 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} = S$$

compute the inverse

$\det(S) = 1 + 1 = 2$

$$S^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A = S \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1}$$

$$e^{tA} = S \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} S^{-1}$$

$$S \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{t} & 0 \\ 0 & e^{t} & e^{t} \\ e^{3t} & 0 & e^{t} \end{bmatrix}$$

multiply first column by e^{3t}
multiply second column by e^{t}
multiply third column by e^{t}

$$e^{tA} = \tfrac{1}{2}\begin{bmatrix} e^{3t} & e^{t} & 0 \\ 0 & e^{t} & e^{t} \\ e^{3t} & 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} e^{3t}+e^{t} & -e^{3t}+e^{t} & e^{3t}-e^{t} \\ 0 & 2e^{t} & 0 \\ e^{3t}-e^{t} & -e^{3t}+e^{t} & e^{3t}+e^{t} \end{bmatrix}$$

Check $e^{0A} = I$

$$\frac{d}{dt} e^{tA} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$