1. **Practice Midterm 1**

**Problem 1.** At what value(s) of $x$ is the following function discontinuous?

\[ f(x) = \begin{cases} 
  x^2 + 4x + 5 & : \text{if } x < -2 \\
  \frac{1}{2}x & : \text{if } -2 \leq x \leq 2 \\
  1 + \sqrt{x-2} & : \text{if } x > 2
\end{cases} \]

- a) -2
- b) 0
- c) -2, 0, and 2
- d) -2 and 0
- e) 2
- f) -2 and 2
- g) 0 and 2
- h) $f$ is continuous everywhere

Since polynomials are cont. in their domain, $f(x)$ is cont. on $(-\infty, -2)$ and $f(x)$ is cont. on $(-2, 2)$.

Since square root functions are cont. where their argument is non-negative, $f(x)$ is cont. on $(2, +\infty)$.

Hence, we need only check continuity at $2$ and $-2$.

\[
\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} x^2 + 4x + 5 = (-2)^2 + 4(-2) + 5 = 1
\]

\[
\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{1}{2}x = \frac{1}{2}(-2) = -1
\]

Hence, $f(x)$ is not cont. at $x = -2$.

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{2}(x) = \frac{1}{2}(2) = 1
\]

\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 1 + \sqrt{x-2} = 1 + \sqrt{2-2} = 1
\]

Since $f(2) = \lim_{x \to 2} f(x)$, then $f$ is cont. at $x = 2$.
Problem 2. The hypotenuse $AB$ of a right triangle $ABC$ remains constant at 5 feet as both legs are changing. One leg, $AC$, is decreasing at the rate of 2 feet per second. In order for the hypotenuse to remain 5 feet, the other leg, $BC$, is increasing. The rate, in square feet per second, at which the area is changing when $AC = 3$ is

a) $\frac{3}{4}$  

b) $\frac{1}{2}$  

c) $\frac{5}{3}$  

d) $\frac{2}{3}$  

e) $\frac{3}{5}$  

g) $\frac{2}{5}$  

h) None of these

What we know:

- $\frac{dy}{dt} = -2$  
- $x^2 + y^2 = 25$  
- $A = \frac{1}{2}xy$

What we want to know:

What is $\frac{dA}{dt}$ when $y = 3$.

Since $A = \frac{1}{2}xy$, then $\frac{dA}{dt} = \frac{1}{2} \left( \frac{dx}{dt}y + x \frac{dy}{dt} \right)$.

So, to find $\frac{dA}{dt}$, we still need to find $\frac{dx}{dt}$ and $x$ when $y = 3$.

If $y = 3$, then $x^2 + 3^2 = 25$. So, $x = 4$.

Since $x^2 + y^2 = 25$, then $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$.

Plugging in what we know: $2(4) \frac{dx}{dt} + 2(3)(-2) = 0$.

Hence, $\frac{dx}{dt} = \frac{3}{2}$.

Now we can find $\frac{dA}{dt} = \frac{1}{2} \left( \frac{3}{2}(3) + (-2)(4) \right) = \frac{-7}{4} \text{ ft}^2 \text{ sec}^{-1}$.  

$\sqrt{5}$
Problem 3
If \( x^2 - xy - y^3 = 13 \), then find \( \frac{dy}{dx} \) evaluated at \((4,1)\).

a) 0
d) \frac{y}{x}
e) -2
f) -1
\( \boxed{g) \frac{1}{2}} \)
h) ?

\[
2x - (x \frac{dy}{dx} + y) - 3y^2 \frac{dy}{dx} = 0
\]

\[
2x - x \frac{dy}{dx} - y - 3y^2 \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} \left(-x - 3y^2\right) = y - 2x
\]

\[
\frac{dy}{dx} = \frac{y - 2x}{-x - 3y^2}
\]

\[
\frac{dy}{dx} = \frac{1 - 2(4)}{-(4) - 3(1)}
\]

\[
\frac{dy}{dx} = 1
\]
Problem 4. What is the slope of the tangent line to \( f(x) = (x)(\cos(x^2)) \) at
\[
x = \sqrt{\frac{\pi}{2}}
\]
(a) \(-\pi\)
(b) \(\pi\)
(c) 0
(d) 1
(e) \(-1\)
(f) \(\frac{1}{2}\)

First, find \( f'(x) \).

\[
f'(x) = x \cdot \frac{d}{dx} (\cos(x^2)) + \frac{d}{dx} (x) \cdot \cos(x^2)
\]
\[
= x \cdot (-\sin(x^2)) \cdot \frac{d}{dx} (x^2) + \cos(x^2)
\]
\[
= x \cdot (-\sin(x^2)) (2x) + \cos(x^2)
\]
\[
= -2x^2 \sin(x^2) + \cos(x^2)
\]

\[
f'(\sqrt{\frac{\pi}{2}}) = -2(\sqrt{\frac{\pi}{2}})^2 \sin(\frac{\pi}{2}) - \cos((\sqrt{\frac{\pi}{2}})^2)
\]
\[
= -2(\frac{\pi}{2}) \sin(\frac{\pi}{2}) - \cos(\frac{\pi}{2})
\]
\[
= -\pi (1) - 0
\]
\[
= -\pi
\]

\[
f(\sqrt{\frac{\pi}{2}}) = \sqrt{\frac{\pi}{2}} \cos((\sqrt{\frac{\pi}{2}})^2) = \sqrt{\frac{\pi}{2}} \cos(\frac{\pi}{2}) = 0
\]

Since \( f'(\sqrt{\frac{\pi}{2}}) \) is the slope of the tangent line and \((\sqrt{\frac{\pi}{2}}, 0)\) is a point on the line, then, using \(x\)-int. form of a line, \(y = -\pi (x - \sqrt{\frac{\pi}{2}})\) is the tangent line.
Problem 5. The function \( f(x) = (x - 3)^{\frac{2}{3}} \) is increasing for what values of \( x \)?

a) \((-\infty, \infty)\)

\(\boxed{b) (3, \infty)}\)

c) nowhere

d) \((-\infty, 3)\)

e) \((0, \infty)\)

f) everywhere except 3

To find where \( f(x) \) is increasing, we need to find where \( f'(x) > 0 \).

\[
f'(x) = \frac{2}{3} \left( x - 3 \right)^{-\frac{1}{3}} \cdot \frac{d}{dx} \left( x - 3 \right)
\]

\[
= \frac{2}{3} \left( x - 3 \right)^{-\frac{1}{3}} \cdot (1)
\]

Where is \( \frac{2}{3} \left( x - 3 \right)^{-\frac{1}{3}} > 0 \)?

\( (x - 3)^{-\frac{1}{3}} > 0 \)

\( \left( \frac{1}{(x - 3)^{\frac{1}{3}}} \right)^3 > (0)^3 \)

\( \frac{1}{x - 3} > 0 \)

Hence \( \frac{1}{x - 3} > 0 \) only when \( x > 3 \).

Thus, \( f'(x) > 0 \) only when \( x > 3 \).
Problem 6.
Use the intermediate value theorem to show that there is a number that is exactly one more than its cube.

We want to find a number $a$ such that $a = a^3 + 1$.

Let $f(x) = x^3 - x + 1$.

$f(-2) = (-2)^3 + (-2) + 1 = -5$

$f(1) = 1^3 - 1 + 1 = 1$

Hence, by the I.V. theorem, there is a number $a$ in the interval $(-2, 1)$ such that $f(a) = 0$.

So $a^3 - a + 1 = 0$

$a = a^3 + 1$.

Thus, we have shown that such an $a$ exists.
Problem 7. Find the value of the limit.

\[
\lim_{x \to 2} \frac{\sqrt{x+7} - 3}{(x-2)(x+1)} = \lim_{x \to 2} \frac{(\sqrt{x+7} - 3)(\sqrt{x+7} + 3)}{(x-2)(x+1)(\sqrt{x+7} + 3)}
\]

\[
= \lim_{x \to 2} \frac{x+7 - 9}{(x-2)(x+1)(\sqrt{x+7} + 3)}
\]

\[
= \lim_{x \to 2} \frac{x-2}{(x-2)(x+1)(\sqrt{x+7} + 3)}
\]

\[
= \lim_{x \to 2} \frac{1}{(x+1)(\sqrt{x+7} + 3)}
\]

\[
= \frac{1}{(2+1)(\sqrt{9} + 3)}
\]

\[
= \frac{1}{3(3+3)}
\]

\[
= \frac{1}{24}
\]
Problem 8. Let \( V \) be the volume of a cylinder having height \( h \) and radius \( r \), and assume that \( h \) and \( r \) vary with time. When the height is 5 in. and is increasing at 0.2 in./s., the radius is 3 in. and is decreasing at 0.1 in./s. How fast is the volume changing at that instant?

**What we know:**

\[
V = \pi r^2 h
\]

\[
\frac{dh}{dt} = 0.2
\]

\[
\frac{dr}{dt} = -0.1
\]

**What we want to find:**

\[
\frac{dV}{dt}
\]

when \( h = 5 \) and \( r = 3 \)

Since \( V = \pi r^2 h \), then

\[
\frac{dV}{dt} = \pi \left( r^2 \frac{dh}{dt} + h \frac{d(r^2)}{dt} \right)
\]

\[
\frac{dV}{dt} = \pi \left( r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \right)
\]

Now plug in all of the known quantities in boxes.

\[
\frac{dV}{dt} = \pi \left( 3^2 \cdot 0.2 + 2(3)(5)(-0.1) \right)
\]

\[
= \pi \left( 1.8 - 3 \right)
\]

\[
= -1.2 \pi \frac{in^3}{sec.}
\]
Problem 9. Suppose \( f(3) = 2 \), \( f'(3) = 5 \), and \( f''(3) = -2 \). Let \( g(x) = [f(x)]^2 \).

Find the value of \( g''(3) \).

\[
\begin{align*}
g(x) &= (f(x))^2 \\
g'(x) &= 2f(x) \cdot f'(x) \\
g''(x) &= 2f(x) \cdot f''(x) + f'(x) \cdot f'(x) \\
g''(x) &= 2f(x) \cdot f''(x) + 2(f'(x))^2 \\
g''(3) &= 2f(3) \cdot f''(3) + 2(f'(3))^2 \\
&= 2(2)(-2) + 2(5)^2 \\
&= -8 + 50 \\
&= 42
\end{align*}
\]

\[g''(3) = 42\]
Problem 10. If \( f(x) = \frac{x}{\tan(x)} \), find \( f'(\frac{\pi}{4}) \). Do not leave any trigonometric functions in your answer.

\[
\begin{align*}
  f(x) &= \frac{x}{\tan(x)} \\
  f'(x) &= \frac{\tan(x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\tan(x))}{(\tan(x))^2} \\
  &= \frac{\tan(x) - x(\sec(x))^2}{(\tan(x))^2} \\
  &= \frac{\sin(x)}{\cos(x)} - x \left( \frac{1}{\cos(x)} \right)^2 \\
  &= \frac{1}{\sqrt{2}} - \frac{\pi}{4} \left( \frac{1}{\sqrt{2}} \right)^2 \\
  &= 1 - \frac{\pi}{4} \left( \sqrt{2} \right)^2 \\
  &= 1 - \frac{\pi}{2}
\end{align*}
\]