Math 104: Taylor series, Limits and l’Hospital’s rule

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Outline

1 Review

2 Manipulating Taylor Series

3 l’Hospital’s rule
Taylor Series

Definition

The **Taylor series** generated by a function $f$ at $x = a$ is

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \ldots
\]
Taylor Series

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\]

To use this formula we need to know the values of ALL the derivatives of \( f \) at a value \( a \).
Tricks to finding Taylor Series

**Problem:** Find the first 3 terms of the Taylor series for
\[ f(x) = \cos(x) \sin(x) \] at \( x = 0 \).
Tricks to finding Taylor Series

**Problem:** Find the first 3 terms of the Taylor series for $f(x) = \cos(x)\sin(x)$ at $x = 0$.

**Method of Solution:** Multiply the Taylor Series for $\sin(x)$ at $x = 0$ by the Taylor Series for $\cos(x)$ at $x = 0$. 
**Problem:** Find the first 3 terms of the Taylor series for 
\( f(x) = \cos(x)\sin(x) \) at \( x = 0 \).

**Method of Solution:** Multiply the Taylor Series for \( \sin(x) \) at \( x = 0 \) by the Taylor Series for \( \cos(x) \) at \( x = 0 \).

**What is the point?** Skip tedious derivative calculations.
The geometric series for $|x| > 1$

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{k=0}^{\infty} x^k
\]

Ways to formally manipulate Series:

1. Substitution
2. Differentiation
3. Integration
Manipulating Series

Definition

The geometric series for $|x| > 1$

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Ways to formally manipulate Series:

1. Substitution
2. Differentiation
3. Integration

The pay off of all of this work with Taylor series will be a greater understanding of Limits and Derivatives
Revisiting Limits

Definition

If for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - a| < \delta$ then $|f(x) - L| < \epsilon$, we say

$$\lim_{x \to a} f(x) = L$$
Revisiting Limits

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If for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that whenever \( |x - a| < \delta \) then \( |f(x) - L| < \epsilon \), we say

\[
\lim_{x \to a} f(x) = L
\]

**The Game:** Choose \( \epsilon > 0 \). Then find \( \delta \) such that whenever \( x \) is within \( \delta \) of \( a \), \( f(x) \) is within \( \epsilon \) of \( L \).
Revisiting Limits

**Definition**

If for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - a| < \delta$ then $|f(x) - L| < \epsilon$, we say

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**The Game:** Choose $\epsilon > 0$. Then find $\delta$ such that whenever $x$ is within $\delta$ of $a$, $f(x)$ is within $\epsilon$ of $L$.

**Examples** Let $f(x) = |x|$. Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} f'(x)$.
The power of Taylor series when finding limits

Use your knowledge of Taylor series to find the following limits:

\[
\lim_{x \to 0} \frac{\sin(x)}{x}
\]

\[
\lim_{x \to 0} \frac{\sin(3x)}{e^x - 1}
\]

\[
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}
\]
Theorem

If

\[ \lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x), \]

Then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \]