Math 104: Applications of Definite Integrals

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Outline

1. Review
2. The Definite Integral as a Tool
3. Arc Length
4. Area In Polar Coordinates
Types of integrals

**Indefinite Integrals** represent families of antiderivatives

$$\int x \, dx = \frac{x^2}{2} + c$$

Indefinite integrals are useful for solving differential equations.
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**Definite Integrals** represent the area under the curve

\[ \int_{0}^{2} x \, dx = 2 \]

Definite integrals are useful for solving problems in Geometry, Physics and Statistics.
Definition of Definite Integral

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + \frac{b - a}{n} i) \frac{b - a}{n} \]
Fundamental theorem of calculus

**Theorem**

Let \( f(x) \) be a continuous function with antiderivative \( G(x) \)

1. \[
\frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x)
\]

2. \[
\int_a^b f(x) \, dx = G(b) - G(a)
\]

The big idea:

\[
\int d\, \cdot \cdot \cdot = \cdot \cdot \cdot
\]
The length of a curve

Let's find the length of a curve by approximating by line segments.
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If $f$ is continuous on the interval $[a, b]$, then the length of the graph of $f$ from $a$ to $b$ is

$$L = \int_{a}^{b} \sqrt{1 + (f'(x))^2}$$
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\[
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\]

Example: Find circumference of the circle \( x^2 + y^2 = 4 \).
To calculate area in Cartesian coordinates we integrate a function of \( y \) with respect to \( dx \) (vertical bands) or we integrate a function of \( x \) with respect to \( dy \) (horizontal bands).
Calculating area in different coordinates

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To calculate area in Polar coordinates we integrate a function of $\frac{1}{2}r^2$ with respect to $d\theta$ (wedges) or we integrate a function of $2\pi r$ with respect to $dr$ (circular bands).
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Exercise: Calculate the area of the disk in three different ways: using wedges, using circular bands and using vertical bands.