Math 104: Power Series and Approximations

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A **Power Series** is a series and a function of the form

\[ P(x) = \sum_{k=0}^{\infty} c_k (x - a)^k = c_1 + c_2(x - a) + c_3(x - a)^2 + \ldots \]

where \( x \) is a variable, the \( c_i \) are constants and we say \( P(x) \) is centered at \( a \).

Let \( R \) be the radius of convergence of \( P(x) \).

\[ R = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right| \]
Definition

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If the radius of convergence of a Taylor series is \( R \), find the radius of convergence of an antiderivative and the derivative.
Taylor’s Formula

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \]

Where \( R_n(x) \) is the error term of order \( n \).

Theorem (Taylor’s Theorem)

Given a Taylor Series \( \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \), if there is a constant \( M \) such that \( |f^{(n+1)}(t)| < M \) for all \( t \) between \( a \) and \( x \), then

\[ |R_n(x)| < M \frac{|x-a|^{n+1}}{(n+1)!} \]
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Uses: Can show Taylor series converges if \( |R_n(x)| \) goes to zero as \( n \) goes to infinity, Can get estimates for functions.
Examples

1. Show that the Maclaurin series for $\cos(x)$ converges to $\cos(x)$ for all $x$ using Taylor’s Theorem.

2. Show that the Maclaurin series for $\frac{1}{1-x}$ converges to $\frac{1}{1-x}$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$ using Taylor’s Theorem.

3. Estimate the error for approximating $e^x$ on $[-2, 2]$ using the first four terms of its Maclaurin Series.

4. Estimate the error for approximating $\cos(x)$ on $[-2\pi, 2\pi]$ using the first four terms of its Maclaurin Series.