MATH 104-004 MIDTERM 2

NAME (PRINTED):

RECITATION TIME:

Please *turn off all electronic devices*. You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**, even on multiple choice or short answer questions—the grading will be based on your work shown as well as the end result. Please **clearly mark** a multiple choice option for each problem. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

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Your signature

Problem	**Score (out of)**
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
Total	(70)

1. (10 pts) Find the length of the curve $y = \int_0^x \sqrt{2t}dt$ from $x = 1$ to $x = 2$.

$$\frac{dy}{dx} = \sqrt{2x}$$

$$\text{Arc length} = \int_1^2 \sqrt{1 + (\sqrt{2x})^2}\, dx$$

$$= \int_1^2 \sqrt{1 + 2x}\, dx$$

let $u = 1 + 2x \Rightarrow du = 2dx$

$$= \int_{x=1}^{x=2} \sqrt{u}\, \frac{1}{2}\, du$$

$$= \frac{1}{2} \left. \frac{u^{3/2}}{3/2} \right|_{x=1}^{x=2} = \left. \frac{1}{3} (1+2x)^{3/2} \right|_1^2$$

$$= \boxed{\frac{1}{3} 5^{3/2} - \frac{1}{3} 3^{3/2}}$$

2. (10 pts) Find the mean of the following probability density function:

$$f(x) = \begin{cases} e^{-x} & : if\ x \geq 0 \\ 0 & : if\ x < 0 \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{\infty} x e^{-x}\, dx$$

By parts: let $du = e^{-x}$ $\quad u = -e^{-x}$

$v = x \quad dv = 1$

$$= \lim_{a\to\infty} -xe^{-x}\Big|_0^a - \lim_{a\to\infty} \int_0^a -e^{-x}\, dx$$

$$= \lim_{a\to\infty} -xe^{-x} - e^{-x}\Big|_0^a$$

$$= \lim_{a\to\infty} -ae^{-a} - e^{-a} - (0 - e^{0})$$

$$= 0 - 0 - (-1)$$

$$= \boxed{1}$$

3. (10 pts) Evaluate the following improper integral

$$\int_{-\infty}^{\infty} \frac{(tan^{-1}(x))^2}{x^2+1} dx$$

$$\int_{-\infty}^{\infty} \frac{(\tan^{-1}(x))^2}{x^2+1} dx = \lim_{a\to\infty} \int_{-a}^{0} \frac{(\tan^{-1}(x))^2}{x^2+1} dx$$

$$+ \lim_{b\to\infty} \int_{0}^{b} \frac{(\tan^{-1}(x))^2}{x^2+1} dx$$

~~tha~~ Use u-sub, let $u = \tan^{-1}(x)$

$$du = \frac{1}{x^2+1} dx$$

$$= \lim_{a\to\infty} \int_{x=-a}^{x=0} u^2\, du + \lim_{b\to\infty} \int_{x=0}^{x=b} u^2\, du$$

$$= \lim_{a\to\infty} \frac{u^3}{3}\Big|_{x=-a}^{x=0} + \lim_{b\to\infty} \frac{u^3}{3}\Big|_{x=0}^{x=b}$$

$$= \lim_{a\to\infty} \frac{1}{3}(\tan^{-1}(x))^3\Big|_{-a}^{0} + \lim_{b\to\infty} \frac{1}{3}(\tan^{-1}(x))^3\Big|_{0}^{b}$$

$$= \frac{1}{3}\left(\frac{\pi}{2}\right)^3 + \frac{1}{3}\left(\frac{\pi}{2}\right)^3$$

$$= \boxed{\frac{\pi^3}{12}}$$

4. (10 pts) Show that the following integral converges or show that it diverges.

$$\int_0^\infty \frac{(tan^{-1}(x))^2}{(2x)^2+1}dx$$

$$\frac{-\pi}{2} \leq tan^{-1}(x) \leq \frac{\pi}{2}$$

$$0 \leq (tan^{-1}(x))^2 \leq \frac{\pi^2}{4}$$

$$0 \leq \frac{(tan^{-1}(x))^2}{(2x)^2+1} \leq \frac{\frac{\pi^2}{4}}{(2x)^2+1}$$

By direct comparison test we can evaluate

$$\int_0^\infty \frac{\frac{\pi^2}{4}}{(2x)^2+1}dx = \lim_{a\to\infty} \frac{\pi^2}{4}\int_0^a \frac{1}{(2x)^2+1}dx$$

let $u = 2x \Rightarrow du = 2dx$

$$= \lim_{a\to\infty} \frac{\pi^2}{4}\int_{x=0}^{x=a} \frac{1}{u^2+1}du$$

$$= \lim_{a\to\infty} \frac{\pi^2}{4}\left(tan^{-1}(u)\right)\Big|_{x=0}^{x=a}$$

$$= \lim_{a\to\infty} \frac{\pi^2}{4}\left(tan^{-1}(2x)\right)\Big|_{x=0}^{x=a}$$

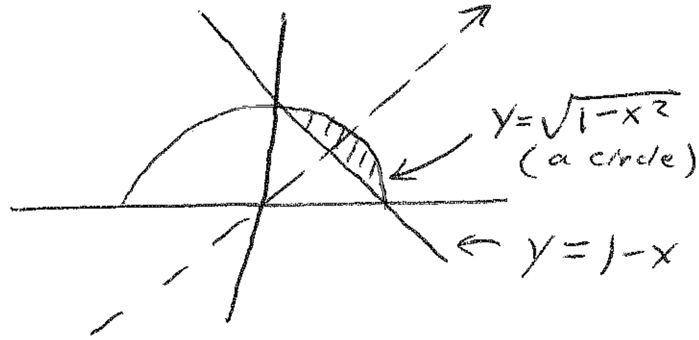
$$= \lim_{a\to\infty} \frac{\pi^2}{4}\left(tan^{-1}(2a) - tan^{-1}(0)\right)$$

$$= \lim_{a\to\infty} \frac{\pi^2}{4}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^3}{8}$$

So, since $\int_0^\infty \frac{\frac{\pi^2}{4}}{(2x)^2+1}dx$ converges, then $\int_0^\infty \frac{(tan^{-1}(x))^2}{(2x)^2+1}dx$ converges.

5. (10 pts) Find the centroid of the region between the curves $y = \sqrt{1-x^2}$ and $y = 1-x$.
Hint: Use symmetry.

Step 1: Draw a pic.



Since the region is symmetric about $y=x$, then $\bar{x}=\bar{y}$

$$\bar{y} = \frac{\int_0^1 (\sqrt{1-x^2})^2 - (1-x)^2 dx}{\int_0^1 \sqrt{1-x^2} - (1-x)\, dx}$$

To find $\int_0^1 \sqrt{1-x^2} - (1-x)\,dx$ notice this is the area of a quarter disk minus the area of , so it is equal to $\frac{\pi}{4} - \frac{1}{2}$

Evaluate $\frac{1}{2}\int_0^1 (\sqrt{1-x^2})^2 - (1-x)^2 dx = \frac{1}{2}\int_0^1 1-x^2-1+2x-x^2\,dx$

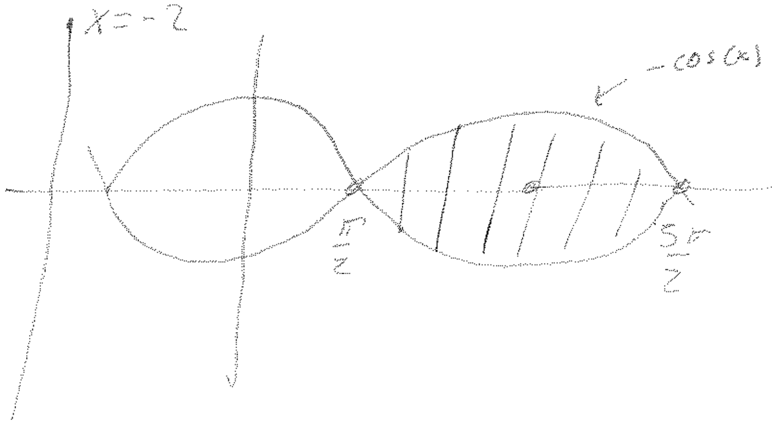
$$= \frac{1}{2}\int_0^1 -2x^2+2x\; dx$$

$$= \frac{1}{2}\left(-\frac{2}{3}x^3 + x^2\right)\Big|_0^1$$

$$= \frac{1}{2}\left(-\frac{2}{3}+1\right) = \left(\frac{1}{3}\right)\frac{1}{2} = \frac{1}{6}$$

So $\bar{x}=\bar{y}= \boxed{\dfrac{\frac{1}{6}}{\frac{\pi}{4}-\frac{1}{2}}}$

7. (10 pts) Write the definite integral representing the volume of the object obtained by rotating the region bounded by $y = cos(x)$ and $y = -cos(x)$ about the line $x = -2$ where the region contains the point $(\pi, 0)$.



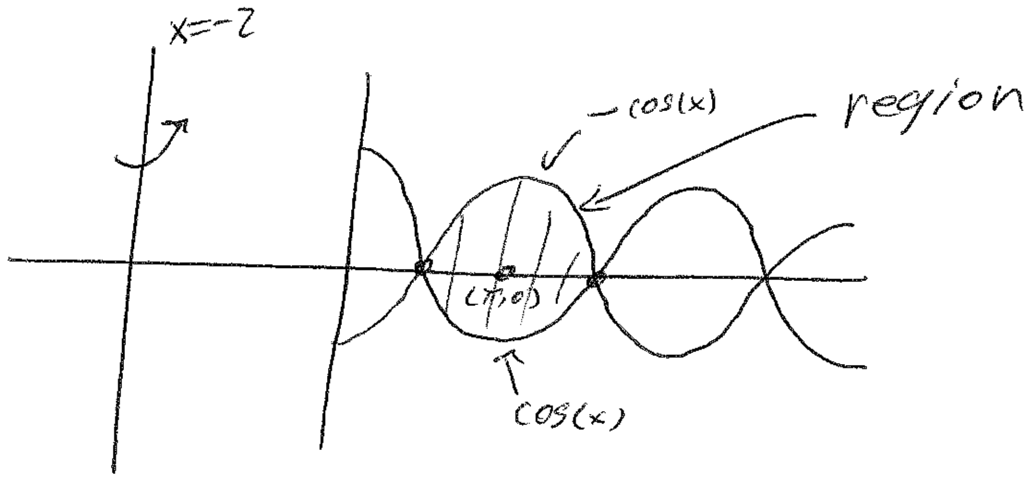
$$Vol = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\pi(2+x)(-\cos(x) - \cos(x))\,dx$$

6. (10 pts) Write the definite integral representing the volume of the object obtained by rotating the region bounded by $y = cos(x)$ and $y = -cos(x)$ about the line $x = -2$ where the region contains the point $(\pi, 0)$.

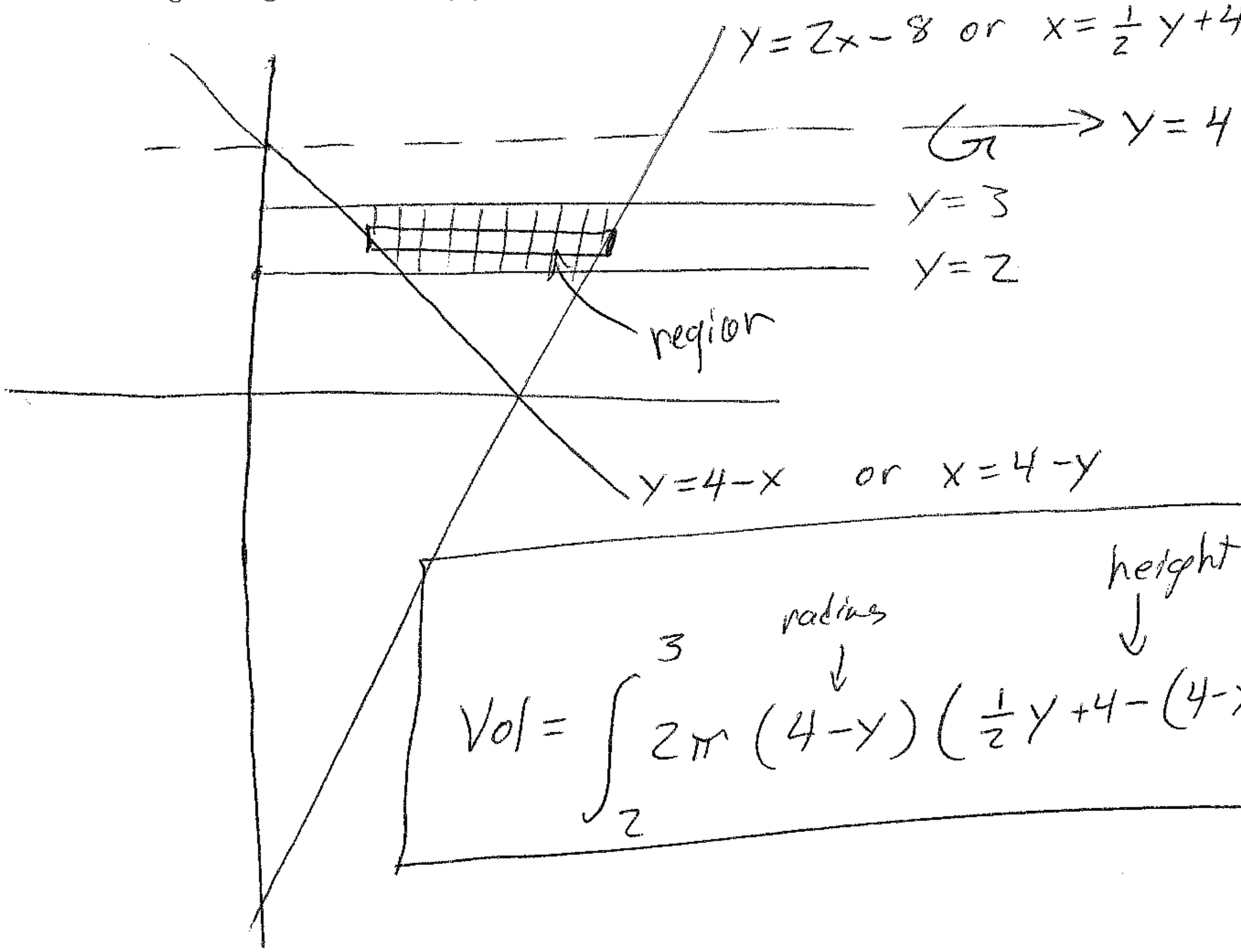
radius

height

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\pi\,(x+2)\,(-\cos(x) - \cos(x))\,dx$$



7. (10 pts) Write the definite integral representing the volume of the object obtained by rotating the region bounded by $y = 3$, $y = 2$, $y = 4 - x$ and $y = 2x - 8$ about the line $y = 4$.



radius

height

$$\boxed{Vol = \int_2^3 2\pi\,(4-y)\left(\tfrac{1}{2}y + 4 - (4-y)\right)dy}$$