Math 104 In Class Practice Midterm 1

Name (printed):

TA:

Recitation Time:

Please *turn off all electronic devices.* You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**, even on multiple choice or short answer questions—the grading will be based on your work shown as well as the end result. Please **clearly mark** a multiple choice option for each problem. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

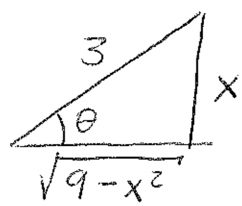
Your signature

Problem	Score (out of)
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
8	(10)
Total	(80)

1. (10 pts) Evaluate

$$\int \frac{x}{\sqrt{9-x^2}}dx$$

Step 1: Make a triangle s.t. one of the sides has length $\sqrt{9-x^2}$



So, $\cos\theta = \dfrac{\sqrt{9-x^2}}{3} \Rightarrow 3\cos\theta = \sqrt{9-x^2}$

$\sin\theta = \dfrac{x}{3} \Rightarrow 3\sin\theta = x$

$\cos\theta\, d\theta = \frac{1}{3}dx \Rightarrow 3\cos\theta\, d\theta = dx$

So, substitute these into

$$\int \frac{x}{\sqrt{9-x^2}}dx = \int \frac{(3\sin\theta)}{(3\cos\theta)}(3\cos\theta\, d\theta)$$

$$= \int 3\sin\theta\, d\theta$$

$$= -3\cos\theta + C$$

$$= -3\,\frac{\sqrt{9-x^2}}{3} + C$$

$$= -\sqrt{9-x^2} + C$$

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or do u-sub letting $u = 9-x^2$

2. (10 pts) Evaluate

$$\int cos^2(x)sin^3(x)dx$$

Recall $\sin^2(x) = 1 - \cos^2(x)$.

$$\int \cos^2(x)(1-\cos^2(x))\sin(x)\,dx$$

$$\int (\cos^2(x) - \cos^4(x))\sin(x)\,dx$$

$$= \int \cos^2(x)\sin(x)\,dx - \int \cos^4(x)\sin(x)\,dx$$

Let $u = \cos(x)$
$du = -\sin(x)\,dx$

$$= \int u^2(-du) - \int u^4(-du)$$

$$= \frac{-u^3}{3} + \frac{u^5}{5} + C$$

$$= \boxed{-\frac{(\cos(x))^3}{3} + \frac{(\cos(x))^5}{5} + C}$$

3. (10 pts) Evaluate

$$\int \frac{x}{(x-1)(x^2+1)}dx$$

Find A, B and C s.t. that

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+1}$$

$$x = A(x^2+1) + (Bx+C)(x-1)$$

Let $x=1$, then

$$1 = A(2) + (B+C)(0) \Rightarrow A = 1/2$$

Let $x = \sqrt{-1} = i$, then

$$i = A(0) + (Bi+C)(i-1)$$

$$i = -B - Bi + Ci - C$$

$$i + 0 = i(C-B) + (-B-C)$$

$$\boxed{C-B=1 \quad \text{and} \quad B+C=0} \Rightarrow \begin{array}{l} 2C=1 \Rightarrow C = 1/2 \\ B = -1/2 \end{array}$$

So $\displaystyle\int \frac{x}{(x-1)(x^2+1)}dx = \int \frac{\frac{1}{2}}{x-1}dx + \int \frac{-1/2\,x + 1/2}{x^2+1}dx$

$$= \frac{1}{2}\ln|x-1| + \int\left(\frac{-1}{2}\right)\frac{x}{x^2+1}dx + \int \frac{1}{2}\frac{1}{x^2+1}dx$$

$$= \boxed{\frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}(x) + C}$$

4. (10 pts)

A vat with 500L of beer contains 4 percent alcohol. Beer with 6 percent alcohol is poured in at a rate of 5L per min. The mixture is pumped out at the same rate. What is the percent of alcohol after one hour?

Let $y(t)$ be the volume of alcohol at time t in Leters.

$$y' = \begin{pmatrix}\text{Rate} \\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{Rate} \\ \text{out}\end{pmatrix} \quad \text{and} \quad y(0) = .04 \times 500 = 20L$$

$$y' = (.06)(5) - \left(\frac{y}{500}\right)(5) \quad \& \quad y(0) = 20$$

$$y' = .3 - \frac{y}{100}$$

$$y' + \frac{1}{100} y = .3$$

Let $v(t) = e^{\int \frac{1}{100} dt} = e^{\frac{t}{100}}$

$$e^{\frac{t}{100}} y' + \frac{1}{100} e^{\frac{t}{100}} y = .3 e^{\frac{t}{100}}$$

$$\frac{d}{dt}\left(e^{\frac{t}{100}} y\right) = .3 e^{\frac{t}{100}}$$

$$e^{\frac{t}{100}} y = \int .3 e^{\frac{t}{100}} dt$$

$$e^{\frac{t}{100}} y = 30 e^{\frac{t}{100}} + C$$

$$\boxed{y = 30 + C e^{\frac{-t}{100}} \quad \& \quad y(0) = 20}$$

Solve for C! $\quad 20 = 30 + C e^{0} \quad C = -10$

$$y(t) = 30 - 10 e^{\frac{-t}{100}}$$

$$y(60) = \boxed{30 - 10 e^{\frac{-60}{100}}}$$

5. (10 pts) Solve the following D.E.

$$x sec^2(y)y' = ln(x)$$

Use separation of variables

$$x \sec^2(y) \frac{dy}{dx} = \ln(x)$$

$$\sec^2(y) \frac{dy}{dx} = \frac{\ln(x)}{x}$$

$$\int \sec^2(y)\, dy = \int \frac{\ln(x)}{x} dx$$

Let $u = \ln(x)$
$du = \frac{1}{x} dx$

$$\tan(y) + C = \int u\, du$$

$$\tan(y) + C = \frac{u^2}{2}$$

$$\tan(y) + C = \frac{1}{2}(\ln(x))^2$$

$$y = \tan^{-1}\left(\frac{1}{2}(\ln(x))^2 + C\right)$$

6. (10 pts) Solve the following D.E.

$$xy' - y = 2xln(x)$$

Rewrite and use the integrating factor method

$$y' - \frac{1}{x} y = 2\ln(x)$$

$$v(x) = e^{\int \frac{-1}{x} dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\frac{1}{x} y' - \frac{1}{x^2} y = \frac{2\ln(x)}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = \frac{2\ln(x)}{x}$$

$$\frac{1}{x} y = \int \frac{2\ln(x)}{x} dx$$

Let $u = \ln(x)$

$du = \frac{1}{x} dx$

$$\frac{1}{x} y = \int 2u\, du$$

$$\frac{1}{x} y = u^2 + C$$

$$\frac{1}{x} y = (\ln(x))^2 + C$$

$$\boxed{y = x(\ln(x))^2 + Cx}$$

7. (10pts) Use Taylor series to find the following limit

$$lim_{x\to 0}\frac{sin(x)-x+\frac{1}{6}x^3}{xcos(x)-x+\frac{1}{2}x^3}$$

Recall $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\lim_{x\to 0}\frac{\sin(x)-x+\frac{1}{6}x^3}{x\cos(x)-x+\frac{1}{2}x^3} = \lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)-x+\frac{1}{6}x^3}{x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)-x+\frac{1}{2}x^3}$$

$$= \lim_{x\to 0}\frac{\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots}{\frac{x^5}{4!}-\frac{x^7}{6!}+\frac{x^9}{8!}-\cdots}$$

$$= \lim_{x\to 0}\frac{x^5\left(\frac{1}{5!}-\frac{x^2}{7!}+\frac{x^4}{9!}-\cdots\right)}{x^5\left(\frac{1}{4!}-\frac{x^2}{6!}+\frac{x^4}{8!}-\cdots\right)}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{5!}-\frac{x^2}{7!}+\frac{x^4}{9!}-\cdots\right)}{\left(\frac{1}{4!}-\frac{x^2}{6!}+\frac{x^4}{8!}-\cdots\right)}$$

$$= \frac{\frac{1}{5!}}{\frac{1}{4!}} = \frac{4!}{5!} = \boxed{\frac{1}{5}}$$

8.(10pts) Using the definition of Taylor series, derive the Taylor series for $f(x) = sin(x)$ at $x = 0$.

Def: $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

$f''''(x) = \sin(x)$ $\quad f''''(0) = 0$

$\vdots$

$$f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is even} \\ (-1)^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \end{cases}$$

T.S. for $f(x) = \sin(x)$ at $x=0$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$