Math 104 In Class Practice Midterm 2

Name (printed):

TA:

Recitation Time:

Please *turn off all electronic devices.* You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**, even on multiple choice or short answer questions—the grading will be based on your work shown as well as the end result. Please **clearly mark** a multiple choice option for each problem. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

Your signature

Problem	Score (out of)
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
8	(10)
Total	(80)

1. (10 pts) Find the mean and median of the following probability density function:

$$f(x) = \begin{cases} \frac{2}{x^3} & : if \ x \geq 1 \\ 0 & : if x < 1 \end{cases}$$

$$\text{Mean} = \mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{1}^{\infty} x \left(\frac{2}{x^3}\right) dx = \lim_{a \to \infty} \int_{1}^{a} \frac{2}{x^2}\, dx$$

$$= \lim_{a \to \infty} \left. \frac{-2}{x} \right|_{1}^{a}$$

$$= \lim_{a \to \infty} \frac{-2}{a} + \frac{2}{1}$$

$$= \boxed{2}$$

$$\text{Median} = m$$

$$\int_{-\infty}^{m} f(x)\, dx = \frac{1}{2}$$

$$\int_{1}^{m} f(x)\, dx = \frac{1}{2}$$

$$\int_{1}^{m} \frac{2}{x^3}\, dx = \frac{1}{2}$$

$$\cancel{\left. \frac{-2}{x} \right|_{1}^{m} = \frac{1}{2}}$$

$$\cancel{\frac{-2}{m} + 2 = \frac{1}{2}}$$

$$\cancel{\frac{-2}{m} = -\frac{3}{2}}$$

$$\cancel{3m = 4}$$

$$\cancel{\boxed{m = \frac{4}{3}}}$$

$$\left. \frac{-1}{x^2} \right|_{1}^{m} = \frac{1}{2}$$

$$-\frac{1}{m^2} + 1 = \frac{1}{2}$$

$$\frac{-1}{m^2} = \frac{-1}{2}$$

$$m^2 = 2$$

$$m = \sqrt{2}$$

2. (10 pts) Show that the following integral converges or show that it diverges.

$$\int_0^\infty \frac{sin^2(x)cos^2(x)}{e^x}dx$$

Since $-1 \le \sin(x) \le 1$, then $0 \le \sin^2(x) \le 1$.

Since $-1 \le \cos(x) \le 1$, then $0 \le \cos^2(x) \le 1$.

Since $0 \le \sin^2(x) \le 1$ and $0 \le \cos^2(x) \le 1$, then

$0 \le \sin^2(x)\cos^2(x) \le 1$.

$$\frac{0}{e^x} \le \frac{\sin^2(x)\cos^2(x)}{e^x} \le \frac{1}{e^x}$$

So, If we show $\int_0^\infty \frac{1}{e^x}dx$ converges, then $\int_0^\infty \frac{\sin^2(x)\cos^2(x)}{e^x}dx$ converges.

Examine $\int_0^\infty \frac{1}{e^x}dx = \lim_{a\to\infty}\int_0^a \frac{1}{e^x}dx = \lim_{a\to\infty} -e^{-x}\Big|_0^a$

$= \lim_{a\to\infty} -e^{-a}+e^0$

$= \boxed{1}$

So, $\int_0^\infty \frac{\sin^2(x)\cos^2(x)}{e^x}dx$ converges.

3. (10 pts) Evaluate the following integral or show that it does not converge.

$$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}dx$$

Note $\frac{1}{\sqrt{|x|}}$ is discontinuous at $x=0$

$$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\,dx = \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{\sqrt{|x|}}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{\sqrt{|x|}}\,dx$$

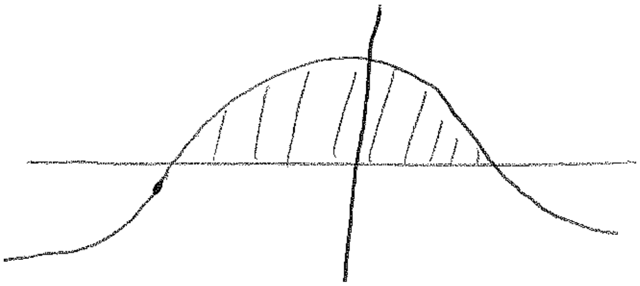
$$= \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{\sqrt{-x}}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^-} -2\sqrt{-x}\,\Big|_{-1}^{a} + \lim_{b\to 0^+} 2\sqrt{x}\,\Big|_{b}^{1}$$

$$= \lim_{a\to 0^-} -2\sqrt{-a} + 2\sqrt{1} + \lim_{b\to 0^+} 2\sqrt{1} - 2\sqrt{b}$$

$$= 0 + 2 + 2 - 0 = \boxed{4}$$

4. (10 pts) Find the centroid of the region bounded by the x-axis and $y = cos(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.



The region is symetric about the y-axis, so $\boxed{\overline{X} = 0}$

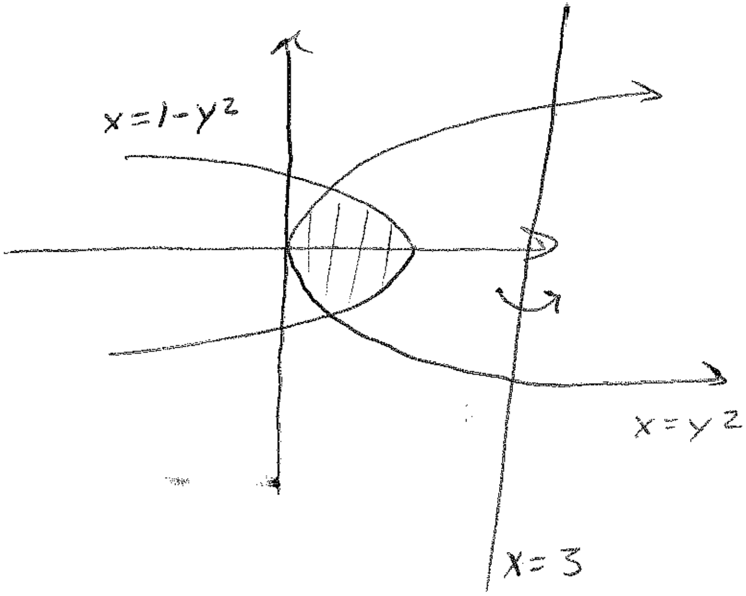
$$\overline{Y} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos^2(x)\,dx}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} + \frac{1}{4}\cos(2x)\,dx}{\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}$$

$$= \frac{\frac{1}{4}x + \frac{1}{8}\sin(2x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}{\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}} = \frac{\left(\frac{\pi}{8}+0\right)-\left(\frac{-\pi}{8}+0\right)}{1-(-1)}$$

$$= \frac{\frac{\pi}{4}}{2}$$

$$= \boxed{\frac{\pi}{8}} = \overline{Y}$$

5. (10 pts) Find the volume of the object obtained by revolving the region bounded by $x = y^2$ and $x = 1 - y^2$ about the line $x = 3$.



Find pts of intersection

$y^2 = 1 - y^2$

$2y^2 = 1$

$y = \pm\sqrt{\frac{1}{2}} = $ [illegible]

Use washer method.

$$Vol = \int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} \pi(3-y^2)^2 - \pi(\underbrace{3-(1-y^2)}_{2+y^2})^2\,dy$$

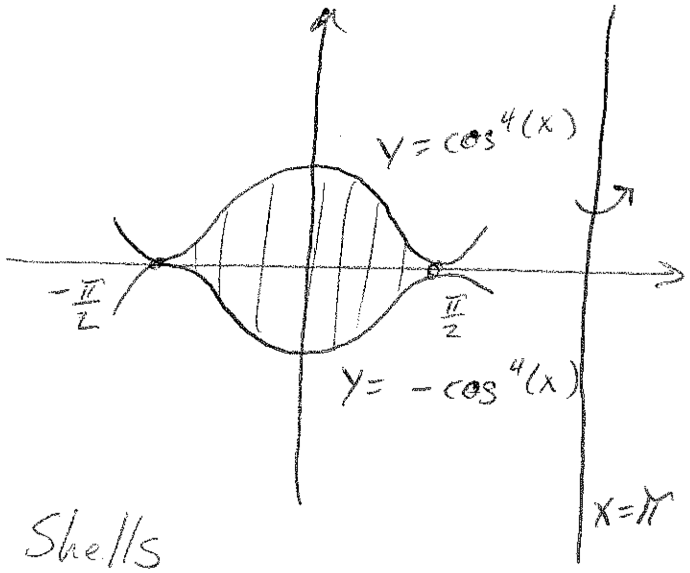
$$= \pi\int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} 9 - 6y^2 + y^4 - 4 - 4y^2 - y^4\,dy$$

$$= \pi\int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} 5 - 10y^2\,dy = \pi\left(5y - \frac{10}{3}y^3\right)\Big|_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}}$$

$$= \pi\left(5\left(\sqrt{\tfrac{1}{2}}\right) - \frac{10}{3}\left(\sqrt{\tfrac{1}{2}}\right)^3\right) - \pi\left(-5\left(\sqrt{\tfrac{1}{2}}\right) + \frac{10}{3}\left(\sqrt{\tfrac{1}{2}}\right)^3\right)$$

$$= 10\pi\sqrt{\tfrac{1}{2}} - \frac{20}{3}\pi\left(\sqrt{\tfrac{1}{2}}\right)^3$$

6. (10 pts) Write the definite integral representing the volume of the object obtained by rotating the region bounded by $y = cos^4(x)$ and $y = -cos^4(x)$ about the line $x = \pi$ where the region contains the point $(0, 0)$.



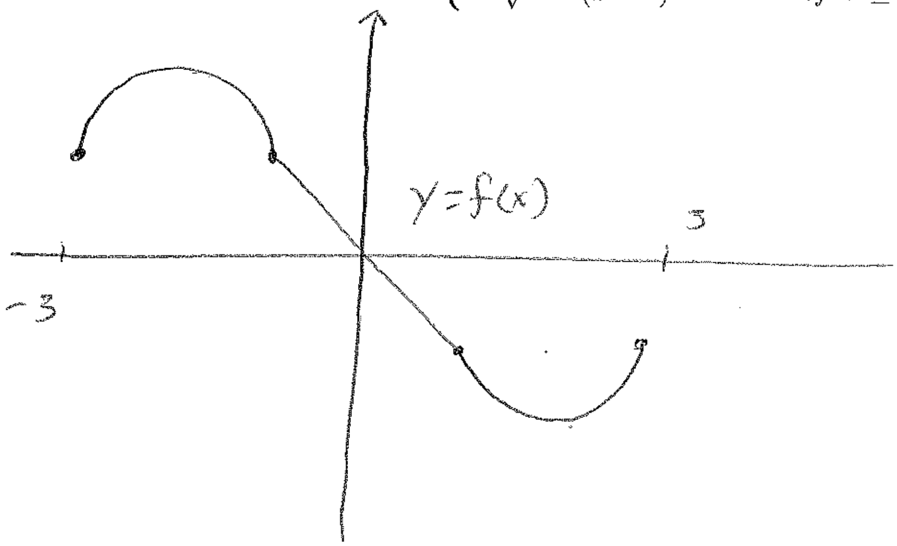
By Shells

$$\text{Vol} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi(\pi - x)\left(\cos^4(x) - (-\cos^4(x))\right)\,dx$$

$$= \boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\pi(\pi - x)\cos^4(x)\,dx}$$

7. (10 pts) Find the arc length of $y = f(x)$ from $x = -3$ to $x = 3$.

$$f(x) = \begin{cases} \sqrt{1-(x+2)^2}+1 & : if \ -3 \le x \ge -1 \\ -x & : if \ -1 \le x \le 1 \\ -\sqrt{1-(x-2)^2}-1 & : if \ 1 \le x \ge 3 \end{cases}$$



y = f(x) is the union of two half circles of radius one and a line segmet from (-1,1) to (1,-1)

So Arc length $= 2\left(\frac{1}{2}(2\pi(1))\right) + \sqrt{(1-(-1))^2+(-1-1)^2}$

$= \boxed{2\pi + \sqrt{2}}$

8. (10 pts) Let $f(x)$ be a solution to the D.E. $(y')^2 = y^2 - 1$. If $f(x) \geq 0$, show that the volume of the object obtained by rotating the region bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ about the x-axis is equal to half the surface area of the object obtained by rotating the curve $y = f(x)$ for $a \leq x \leq b$ about the x-axis.

Since $f(x)$ is a solution to $(y')^2 = y^2 - 1$,

then $(f'(x))^2 = (f(x))^2 - 1$.

Since $f(x) \geq 0$ $\quad f(x) = \sqrt{1 + (f'(x))^2}$

Look at the formula for surface area

$$SA = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_a^b 2\pi f(x) f(x)\, dx$$

$$= 2 \int_a^b \pi (f(x))^2\, dx$$

$= 2$ (volume of the solid)

So Volume of the solid $= \frac{1}{2}$ (Surface area of surface)