Outline

1 Review of Last Time

2 linear Independence
Review of last time

1. Transpose of a matrix
2. Special types of matrices
3. Matrix properties
4. Row-echelon and reduced row echelon form
5. Solving linear systems using Gaussian and Gauss-Jordan elimination
Echelon Forms

Definition

A matrix is in **row-echelon form** if

1. Any row consisting of all zeros is at the bottom of the matrix.
2. For all non-zero rows the leading entry must be a one. This is called the **pivot**.
3. In consecutive rows the pivot in the lower row appears to the right of the pivot in the higher row.

Definition

A matrix is in **reduced row-echelon form** if it is in row-echelon form and every pivot is the only non-zero entry in its column.
Row Operations

We will be applying row operations to augmented matrices to find solutions to linear equations. This is called \textit{Gaussian} or \textit{Gauss-Jordan} elimination.

Here are the row operations:

1. Multiply a row by a number.
2. Switch rows.
3. Add a multiple of one row to another.
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**Key Fact:** If you alter an augmented matrix by row operations you preserve the set of solutions to the linear system.
Today’s Goals

1. Be able to use rank of a matrix to determine if vectors are linearly independent.

2. Be able to use rank of an augmented matrix to determine consistency or inconsistency of a system.
Linear Independence

Definition

Let $v_1, ..., v_m$ be vectors in $\mathbb{R}^n$. The set $S = \{v_1, ..., v_m\}$ is **linearly independent** if $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ implies $c_1 = c_2 = ... = c_n = 0$.

If there exists a non trivial solution to $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ we say the set $S$ is linearly dependant.
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**Example**: Are the following vectors linearly independent?

$< 1, 2, 1 >, < 1, 1, 0 >, < 1, 0, 1 >$
**Definition**

Let $A$ be an $m \times n$ matrix. The **rank** of $A$ is the maximal number of linearly independent row vectors.

**Definition**

(Pragmatic)

Let $A$ be an $m \times n$ matrix and $B$ be its row-echelon form. The **rank** of $A$ is the number of pivots of $B$. 
**Definition**

Let $A$ be an $m \times n$ matrix. The **rank** of $A$ is the maximal number of linearly independent row vectors.

**Example**  What is the rank of the following matrix.

\[
\begin{pmatrix}
2 & 0 & 1 & -1 \\
0 & 1 & 2 & 1 \\
2 & -1 & -1 & -2
\end{pmatrix}
\]
Determining Linear independence Using Matrices

How to find if $m$ vectors are linearly independent:

1. Make the vectors the rows of a $m \times n$ matrix (where the vectors are of size $n$)
2. Find the rank of the matrix.
3. If the rank is $m$ then the vectors are linearly independent. If the rank is less than $m$, then the vectors are linearly dependant.
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**Example:** Are the following vectors linearly independent?

\[
\langle -2, 0, 4, 1 \rangle, \langle 0, 0, 1, -1 \rangle, \langle 0, 1, 0, 1 \rangle, \langle 3, 2, -3, 0 \rangle
\]
Determining Consistency

Given the linear system $Ax = B$ and the augmented matrix $(A|B)$.

1. If $\text{rank}(A) = \text{rank}(A|B) =$ the number of rows in $x$, then the system has a unique solution.

2. If $\text{rank}(A) = \text{rank}(A|B) <$ the number of rows in $x$, then the system has $\infty$-many solutions.

3. If $\text{rank}(A) < \text{rank}(A|B)$, then the system is inconsistent.