Outline

1. Review of Last Time
2. Properties of Determinants
3. Matrix Inverse
4. Properties of Inverses
5. Solving a Linear System Using Inverses
Review of last time

1. How to find determinants using cofactor expansion.
2. How to find determinants using row operations.
Definition of Arbitrary Determinant

Let \( A = (a_{ij})_{n \times n} \) be an \( n \times n \) matrix.

The cofactor expansion of \( A \) along the \( i \)th row is

\[
det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}
\]

The cofactor expansion of \( A \) along the \( j \)th column is

\[
det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}
\]
Using Elementary Row Operations to Find the Determinant

Suppose $B$ is obtained from $A$ by:

1. multiplying a row by a non-zero scalar $c$, then $\det(A) = \frac{1}{c} \det(B)$.
2. switching rows, then $\det(A) = -\det(B)$.
3. adding a multiple of one row to another row, then $\det(A) = \det(B)$.
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Using this idea we can quickly find determinants by row-reducing to triangular form.
Properties of Determinants

Theorem

If elementary row or column operations lead to one of the following conditions, then the determinant is zero.

1. an entire row (or column) consists of zeros.
2. one row (or column) is a multiple of another row (or column).
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Let $A$ and $B$ be $n \times n$ matrices and $c$ be a scalar.

1. $\det(AB) = \det(A)\det(B)$
2. $\det(cA) = c^n\det(A)$
3. $\det(A^T) = \det(A)$
Be able to find the inverse of a matrix or show it has no inverse.

Know the properties of inverses.

Be able to solve systems of linear equations using matrices.
Definition

An $n \times n$ matrix $A$ is **invertible** if there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n.$$ 

In this case, $B$ is the **inverse** of $A$. 
Matrix Inverse

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1. NOT every matrix is invertible.
2. A matrix that is not invertible is called **singular**.
3. If \( A \) is invertible, its inverse is denoted \( A^{-1} \).
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**Example:** check the following matrices are inverses of each other.

$$\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} -7 & 2 \\ 4 & -1 \end{pmatrix}$$
If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is a \( 2 \times 2 \) matrix and \( \det(A) \neq 0 \), then

\[
A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]
Matrix Inverse

A 2 × 2 Matrix Inverse Formula

If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is a 2 × 2 matrix and \( \det(A) \neq 0 \), then

\[
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\]

**Exercise:** Prove the above statement.
Inverses of Arbitrary $n \times n$ Matrices

How to find the inverse of an arbitrary $n \times n$ matrix $A$.

1. Form the augmented $n \times 2n$ matrix $[A|I_n]$.
2. Find the reduced row echelon form of $[A|I_n]$.
3. If $\text{rank}(A) < n$ then $A$ is not invertible.
4. If $\text{rank}(A) = n$, then the RREF form of the augmented matrix is $[I_n|A^{-1}]$. 
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Find the inverse of

\[
\begin{pmatrix}
-1 & 3 & 0 \\
1 & -2 & 1 \\
0 & 1 & 2
\end{pmatrix}
\]
Properties of Inverses

1. \((A^{-1})^{-1} = A\)
2. \((cA)^{-1} = \frac{1}{c} A^{-1}\)
3. \((AB)^{-1} = B^{-1} A^{-1}\)
4. \((A^T)^{-1} = (A^{-1})^T\)
5. \(det(A^{-1}) = \frac{1}{det(A)}\)
6. A is invertible if and only if \(det(A) \neq 0\)
Let $A$ be invertible and $Ax = B$ be a linear system, then the solution to the linear system is given by

$$x = A^{-1}B$$
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**Example:** Solve the following linear system using inverses.

$$x + z = -4$$

$$x + y + z = 0$$

$$5x - y = 6$$