Today’s Goals

Understand the form of solutions to the following types of higher order, linear differential equations

1. Initial Value Problems
2. Boundary Value Problems
3. Homogeneous and Nonhomogeneous Equations.
A Few Famous Differential Equations

1. Einstein’s field equation in general relativity
2. The Navier-Stokes equations in fluid dynamics
3. Verhulst equation - biological population growth
4. The Black-Scholes PDE - models financial markets
Higher Order Initial Value Problems

Definition

For a linear differential equation, an **nth-order initial value problem** (IVP) is

\[
\text{Solve : } a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)
\]

Subject to: \(y(x_0) = y_0, \ y'(x_0) = y_1, \ \ldots, y^{(n-1)}(x_0) = y_{n-1}\)
Existence and Uniqueness

**Theorem**

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$, and $g(x)$ be continuous on and interval $I$, and let $a_n(x) \neq 0$ for every $x$ in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem exists on the interval and is unique.
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Example: Does the following IVP have a unique solution? If so, on what intervals?

$y''' + y'' - y' - y = 9$ with $y(2) = 0$, $y'(2) = 0$ and $y''(2) = 0$
Boundary Value Problem

Definition

For a linear differential equation, an **nth-order boundary value problem** (BVP) is

\[
\begin{align*}
\text{Solve: } & \quad a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \\
\text{Subject to } & \quad n \text{ equations that specify the value of } y \text{ and its derivatives at different points (called boundary conditions).}
\end{align*}
\]
Boundary Value Problem

Definition

For a linear differential equation, an **nth-order boundary value problem** (BVP) is

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \]

Subject to \( n \) equations that specify the value of \( y \) and its derivatives at different points (called **boundary conditions**).

**Question:** What are the possible boundary conditions for a second order linear D.E.
One, Many or No Solutions

A BVP may have one, $\infty$-many, or no solutions.

**Example:** $x'' + 16x = 0$
Homogeneous and Nonhomogeneous

Definition

An nth-order differential equation of the following form is said to be **homogeneous**. Otherwise we say the equation is **nonhomogeneous**.

\[
Solve: \quad a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1(x) \frac{dy}{dx} + a_0(x)y = 0
\]
Theorem

(The Superposition Principle) Let \( y_1, y_2, \ldots, y_k \) be solutions to a homogeneous nth-order differential equation on an interval \( I \). Then any linear combination

\[
y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_k y_k(x)
\]

is also a solution, where \( c_1, c_2, \ldots, c_k \) are constants.
A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is **linearly dependent** on an interval $I$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

for every $x$ in the interval. A set of functions that is not linearly dependent is said to be **Linearly Independent**.
The Wronskian

Definition

Suppose each of the functions \( f_1(x), f_2(x), \ldots, f_n(x) \) possess at least \( n - 1 \) derivatives. The determinant

\[
W(f_1, f_2, \ldots, f_n) = \begin{vmatrix}
  f_1 & f_2 & \ldots & f_n \\
  f'_1 & f'_2 & \ldots & f'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)}
\end{vmatrix}
\]

is called the Wronskian of the functions.
Theorem

Let \( y_1, y_2, \ldots, y_n \) be \( n \) solutions to a homogeneous linear \( n \)th-order differential equation on an interval \( I \). The set of solutions is \textbf{linearly independent} on \( I \) if and only if \( W(y_1, y_2, \ldots, y_n) \neq 0 \) for every \( x \) in the interval. If the solutions \( y_1, y_2, \ldots, y_n \) are linearly independent they are said to be a \textbf{fundamental set of solutions}.

Note: There always exists a fundamental set of solutions to an \( n \)th-order linear homogeneous differential equation on an interval \( I \).
General Solution

Theorem

Let \( y_1, y_2, \ldots, y_n \) be a fundamental set of solutions to an \( n \)th-order linear homogeneous differential equation on an interval \( I \). Then the general solution of the equation on the interval is

\[
y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)
\]

where the \( c_i \) are arbitrary constants.
Theorem

Let $y_p$ be any particular solution of the nonhomogeneous linear $n$th-order differential equation on an interval $I$. Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions to the associated homogeneous differential equation. Then the general solution to the nonhomogeneous equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) + y_p$$

where the $c_i$ are arbitrary constants.
Superposition Principle for Nonhomogeneous Equations

Theorem

Suppose \( y_{p_i} \) denotes a particular solution to the differential equation

\[
a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x) \frac{dy}{dx} + a_0(x)y = g_i(x)
\]

Where \( i = 1, 2, \ldots, k \). Then \( y_p = y_{p_1} + y_{p_2} + \ldots + y_{p_k} \) is a particular solution of

\[
a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x) \frac{dy}{dx} + a_0(x)y =
\]

\[
g_1(x) + g_2(x) + \ldots + g_k(x)
\]